Weighted Quasi-Arithmetic Means and Invariance of Types 1, 2, and 3

Alan Horwitz
25 Yearsley Mill Rd.
Penn State Brandywine
Media, PA 19063 USA
alh4@psu.edu

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Abstract

Let m_n and m_{n-1} be an n mean and an $n-1$ mean, respectively, $n \geq 3$. If $\hat{x} = (x_1, ..., x_n)$, let $\pi_{\neq j}\hat{x} = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)$.

m_{n-1} and m_n are said to form a type 1 invariant pair if

$m_n(m_{n-1}(\pi_{\neq 1}\hat{x}), m_{n-1}(\pi_{\neq 2}\hat{x}), ..., m_{n-1}(\pi_{\neq n}\hat{x})) = m_n(\hat{x})$ for all $\hat{x} \in \Re^n$.

m_{n-1} and m_n are said to form a type 2 invariant pair if $m_n(\hat{x}, m_{n-1}(\hat{x})) = m_{n-1}(\hat{x})$ for all $\hat{x} \in \Re_+^{n-1}$.

If $\hat{x} = (x_1, ..., x_{n-1})$, let $\pi_{=j}\hat{x} = (x_1, ..., x_{j-1}, x_j, x_j, x_{j+1}, ..., x_{n-1}) \in R_+^n$.

m_{n-1} and m_n are said to form a type 3 invariant pair if

$m_{n-1}(m_n(\pi_{=1}\hat{x}), ..., m_n(\pi_{=n-1}\hat{x})) = m_{n-1}(\hat{x})$ for all $\hat{x} \in \Re_+^{n-1}$. Let $m_{h,w,n}(a_1, ..., a_n) = h^{-1}\left(\dfrac{\sum_{k=1}^{n} w(a_k)h(a_k)}{\sum_{k=1}^{n} w(a_k)}\right)$, where $h(x)$ is continuous and monotone, and $w(x)$ is continuous and positive, on $(0, \infty)$ denote the family of weighted quasi–arithmetic means in n variables.

We prove that if $m_{h,w,n}$ and $m_{h,w,n-1}$ form a type 1 or type 3 invariant pair, then $m_{h,w,n}$ and $m_{h,w,n-1}$ are quasi–arithmetic means. The method of proof involves deriving equations for certain partial derivatives of order 3 of $m_{h,w,n}$ on the diagonal of $\Re_+^n$. The proof also requires an equation relating certain partial derivatives of order 3 for type 1 or type 3 invariant pairs of means. We also show that any pair of weighted quasi–arithmetic means $m_{h,w,n}$ and $m_{h,w,n-1}$ form a type 2 invariant pair.

1 Introduction

Let $\Re = (\infty, \infty), \Re_+^n = \{(a_1, ..., a_n) \in \Re^n : a_i > 0 \; \forall i\}$; We define a *mean*, m, in n variables(n mean for short as in [2]) to be a function on $\Re_+^n$ with

$\min(a_1, ..., a_n) \leq m(a_1, ..., a_n) \leq \max(a_1, ..., a_n)$.

m is called *symmetric* if $m(\pi(a_1, ..., a_n)) = m(a_1, ..., a_n)$ for any permutation π. It follows immediately that m satisfies the reflexive condition $m(a, ..., a) = a$ for any $a \in \Re^+$. Note that sometimes the weaker reflexive condition is given as the definition of a mean. Let m_n and m_{n-1} be an n mean and an $n-1$ mean, respectively. In [1] the author introduced the notions of type 1 and type 2 invariance, which are defined as follows.

Definition 1 *Let m_n and m_{n-1} be an n mean and an $n-1$ mean, respectively, $n \geq 3$. If $\hat{x} = (x_1, ..., x_n)$, let $\pi_{\neq j}\hat{x} = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_n) \in \Re_+^{n-1}$. m_{n-1} and m_n are said to form a type 1 invariant pair, written $(m_{n-1}, m_n) \in T_1$, if*

$$m_n(m_{n-1}(\pi_{\neq 1}\hat{x}), m_{n-1}(\pi_{\neq 2}\hat{x}), ..., m_{n-1}(\pi_{\neq n}\hat{x})) = m_n(\hat{x})$$

for all $\hat{x} \in \Re_+^n$. That is, $(m_{n-1}, m_n) \in T_1$ if
$m_n(m_{n-1}(x_2, ..., x_n), ..., m_{n-1}(x_1, ..., x_{n-1})) = m_n(x_1, ..., x_n)$.

For example, $m(a, b)$ and $M(a, b, c)$ form a type 1 invariant pair if $M(m(a, c), m(a, b), m(b, c)) = M(a, b, c)$ for all $(a, b, c) \in \Re_+^3$. In [2], if $(m_{n-1}, m_n) \in T_1$, then m_n is called a β invariant extension of m_{n-1}.

Definition 2 *Let m_n and m_{n-1} be an n mean and an $n-1$ mean, respectively, $n \geq 3$. m_{n-1} and m_n are said to form a type 2 invariant pair, written $(m_{n-1}, m_n) \in T_2$, if*

$$m_n(x_1, ..., x_{n-1}, m_{n-1}(x_1, ..., x_{n-1})) = m_{n-1}(x_1, ..., x_{n-1})$$

for all $(x_1, ..., x_{n-1}) \in \Re_+^{n-1}$.

For example, $m(a, b)$ and $M(a, b, c)$ form a type 2 invariant pair if $M(a, b, m(a, b)) = m(a, b)$ for all $(a, b, c) \in \Re_+^2$.

Numerous results were proven in [1] for type 1 and type 2 invariance for means in two and three variables. See [2] for additional results for type 1and type 2 invariance for means in n variables. In this paper we also introduce a third type of invariance, which is similar, though somewhat different, than the other two types.

Definition 3 *Let m_n and m_{n-1} be an n mean and an $n-1$ mean, respectively, $n \geq 3$. If $\hat{x} = (x_1, ..., x_{n-1})$, let $\pi_{=j}\hat{x} = (x_1, ..., x_{j-1}, x_j, x_j, x_{j+1}, ..., x_{n-1}) \in \Re_+^n$. m_{n-1} and m_n are said to form a type 3 invariant pair, written $(m_{n-1}, m_n) \in T_3$, if*

$$m_{n-1}(m_n(\pi_{=1}\hat{x}), ..., m_n(\pi_{=n-1}\hat{x})) = m_{n-1}(\hat{x})$$

for all $\hat{x} \in \Re_+^{n-1}$. That is, $(m_{n-1}, m_n) \in T_3$ if
$m_{n-1}(m_n(x_1, x_1, x_2, ..., x_{n-1}), ..., m_n(x_1, x_2, ..., x_{n-1}, x_{n-1})) =$
$m_{n-1}(x_1, ..., x_{n-1})$.

For example, $m(a,b,c)$ and $M(a,b,c,d)$ form a type 3 invariant pair if $m(M(a,a,b,c), M(a,b,b,c), M(a,b,c,c)) = m(a,b,c)$ for all $(a,b,c) \in \Re_+^3$. If $(m_{n-1}, m_n) \in T_k$, $k = 1, 2$, or 3, we shall also sometimes say that m_n is a *type k invariant extension* of m_{n-1}, or that m_{n-1} is a *type k invariant reduction* of m_n.

It is easy to show(see Theorem 1) that $(m_{h,n-1}, m_{h,n}) \in T_k, k = 1,2,3$, where $m_{h,n}(a_1, ..., a_n) = h^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} h(a_k)\right)$ are the quasi–arithmetic means, $h(x)$ a given function continuous and monotone on $(0, \infty)$. One might then ask if the same holds for the **weighted** quasi–arithmetic means $m_{h,w,n}(a_1, ..., a_n) = h^{-1}\left(\frac{\sum_{k=1}^{n} w(a_k)h(a_k)}{\sum_{k=1}^{n} w(a_k)}\right)$, where $w(x)$ is continuous and positive on $(0, \infty)$. In [1], the author stated(without proof) that if one uses the **same** h and w, then the only pairs of weighted quasi–arithmetic means in two and three variables which are type 1 invariant are the quasi–arithmetic means. However, the proof of this does not appear to be as short or simple as this author had originally thought. In section 2 we supply a proof(Theorem 2) that $(m_{h,w,n}, m_{h,w,n-1}) \in T_1 \iff m_{n-1}$ and m_n are quasi–arithmetic means. We also prove (Theorem 4) the same result for type 1 invariance, but where we asume that $n \geq 4$. Unlike the situation for type 1 or type 3 invariance, it is easy to prove(see Theorem 3 in 2) that any pair of weighted quasi–arithmetic means are type 2 invariant. The proofs of Theorems 2 and Theorem 4 require the partial derivatives of $m_{h,w,n}$ and of $m_{h,w,n-1}$ on the diagonal(all coordinates equal), through the third order. The formulas for these partial derivatives in terms of w, h, and n are given in section 2(Proposition 1). We also require equations relating the partial derivatives on the diagonal, through the third order, of any pair, (m_{n-1}, m_n) of type 1 or type 3 invariant means. We also give such an equation for $(m_{n-1}, m_n) \in T_2$, though we do not use that equation to prove anything. These equations are given in section 2(see Propositions 1, 2, 3, and 4). The third order equations are different for each type of invariance, but the second order equations are identical. More specifically, if $(m_{n-1}, m_n) \in T_k, k = 1, 2$, or 3, then on the diagonal, m_{n-1} and m_n satisfy the second order equation$\frac{\partial^2 m_n}{\partial x_1^2} = \frac{(n-1)^3}{n^2(n-2)}\frac{\partial^2 m_{n-1}}{\partial x_1^2}$. The proofs of Propositions 1, 2, 3, and 4 just involve the product and chain rules, along with Lemma 2 and Theorem 7(see the Appendix), but they are rather long and tedious. Hence we leave their proofs to the Appendix. The dedicated reader may try to follow all of the details, or perhaps come up with a shorter and less tedious proof. The equations given in 2, 3, and 4 are of interest in their own right and could be used to prove that certain other pairs of means (m_{n-1}, m_n) are not invariant pairs of a certain type.

2 Main Results

For brevity of notation, we leave off the dependence of the weighted quasi–arithmetic means, $m_{h,w}$, on n, h, and w.

Proposition 1 *For fixed $n \geq 2$, let $x^{[n]} = (x, ..., x) \in \Re_+^n$. Let $m(x_1, ..., x_n) = h^{-1}\left(\frac{\sum_{k=1}^{n} w(x_k)h(x_k)}{\sum_{k=1}^{n} w(x_k)}\right)$ be a weighted quasi–arithmetic mean, where $w(x)$ is twice differentiable and positive on $(0, \infty)$, and $h(x)$ is three times differentiable and monotone on $(0, \infty)$. Then*

$$\frac{\partial^2 m}{\partial x_1^2}(x^{[n]}) = \frac{n-1}{n^2}\frac{2h'w'+wh''}{h'w}, \tag{2.1}$$

$$\frac{\partial^2 m}{\partial x_1 \partial x_2}(x^{[n]}) = \frac{-2h'w'-wh''}{n^2 h'w}, \tag{2.2}$$

$$\frac{\partial^3 m}{\partial x_1^3}(x^{[n]}) = \frac{n-1}{n^3}\frac{3n(h')^2ww''+3(n-2)ww'h'h''+(n+1)w^2h'h'''-6(h')^2(w')^2-3w^2(h'')^2}{(h')^2w^2}, \tag{2.3}$$

and

$$\frac{\partial^3 m}{\partial x_1^2 \partial x_2}(x^{[n]}) = \frac{2(3-n)(h')^2(w')^2+3(2-n)h'h''ww'-nww''(h')^2+(3-n)(w)^2(h'')^2-h'(w)^2h'''}{n^3(h')^2w^2}. \tag{2.4}$$

Proposition 2 *Suppose that the n mean, m_n, is type 1 invariant with respect to the $n-1$ mean, $m_{n-1}, n \geq 3$, where m_n and m_{n-1} are three times differentiable and symmetric means on $\Re_+^n$ and on $\Re_+^{n-1}$, respectively. Then for all $x > 0$,*

$$\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) = \frac{n^2(n-2)}{(n-1)^3}\frac{\partial^2 m_n}{\partial x_1^2}\left(x^{[n]}\right) \tag{2.5}$$

and

$$\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]}) = \frac{n(n^3-3n^2+3n-3)}{(n-1)^4}\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}) \tag{2.6}$$
$$-\frac{3n}{(n-1)^3}\frac{\partial^3 m_n}{\partial x_1^2 \partial x_2}(x^{[n]}) - \frac{3n^3(n-2)}{(n-1)^5}\left(\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\right)^2.$$

Proposition 3 *Let $x^{[n]} = (x, ..., x) \in \Re_+^n$ and suppose that the n mean, m_n, is type 2 invariant with respect to the $n-1$ mean, $m_{n-1}, n \geq 3$, where m_n and m_{n-1} are three times differentiable and symmetric on $\Re_+^n$ and on $\Re_+^{n-1}$, respectively. Then for all $x > 0$*

$$\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) = \frac{(n-1)^3}{n^2(n-2)}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})$$

and

$$\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]}) = \frac{n^2(n^2-3n+3)}{(n-1)^4}\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}) + \frac{3n^2}{(n-1)^3}\frac{\partial^3 m_n}{\partial x_1^2 \partial x_2}(x^{[n]}).$$

Proposition 4 *Let* $x^{[n]} = (x, ..., x) \in \Re_+^n$ *and suppose that the* n *mean,* m_n*, is type 3 invariant with respect to the* $n-1$ *mean,* $m_{n-1}, n \geq 3$*, where* m_n *and* m_{n-1} *are three times differentiable and symmetric on* $\Re_+^n$ *and on* $\Re_+^{n-1}$*, respectively. Then for all* $x > 0$

$$\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) = \frac{(n-1)^3}{n^2(n-2)} \frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) \tag{2.7}$$

and

$$\frac{n}{n-1}\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}) + \frac{6}{n-1}\frac{\partial^3 m_n}{\partial x_1^2 \partial x_2}(x^{[n]}) - \frac{n^3-n-6}{n^3}\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]}) + \frac{3(n^2+n-6)}{n^3}\frac{\partial^3 m_{n-1}}{\partial x_1^2 \partial x_2}(x^{[n-1]}) + \frac{(n-3)(n-2)(n+2)}{n^3}\frac{\partial^3 m_{n-1}}{\partial x_1 \partial x_2 \partial x_3}(x^{[n-1]}) + \frac{3(n-3)}{n(n-1)}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) = 0. \tag{2.8}$$

Let ϕ be a continuous, monotonic function on $(0, \infty)$ and let $m(a_1, ..., a_n)$ be an n mean. Define another n mean, m_ϕ, by

$$m^\phi(a_1, ..., a_n) = \phi^{-1}(m(\phi(a_1), ..., \phi(a_n))).$$

Lemma 1 $(m_{n-1}, m_n) \in T_k \Leftrightarrow (m_{n-1}^\phi, m_n^\phi) \in T_k, k = 1,2,3$

Proof. Since the inverse of a continuous, monotonic function on $(0, \infty)$ is also monotonic a continuous, monotonic function on $(0, \infty)$, it suffices to prove that $(m_{n-1}, m_n) \in T_k \Rightarrow (m_{n-1}^\phi, m_n^\phi) \in T_k$.

For Type 1: Suppose that $(m_{n-1}, m_n) \in T_1$.
$m_n^\phi(m_{n-1}^\phi(x_2, ..., x_n), ..., m_{n-1}^\phi(x_1, ..., x_{n-1})) =$
$\phi^{-1}(m_n(\phi(m_{n-1}^\phi(x_2, ..., x_n)), \phi ..., \phi(m_{n-1}^\phi(x_1, ..., x_{n-1})))) =$
$\phi^{-1}(m_n(m_{n-1}(\phi(x_2), ..., \phi(x_n)), ..., m_{n-1}(\phi(x_1), ..., \phi(x_{n-1})))) =$
$\phi^{-1}(m_n(\phi(x_1), ..., \phi(x_n))) = m_n^\phi(x_1, ..., x_n)$, which implies that
$(m_{n-1}^\phi, m_n^\phi) \in T_1$

For Type 2: Suppose that $(m_{n-1}, m_n) \in T_2$.
$m_n^\phi(x_1, ..., x_{n-1}, m_{n-1}^\phi(x_1, ..., x_{n-1})) =$
$m_n^\phi(x_1, ..., x_{n-1}, \phi^{-1}(m_{n-1}(\phi(x_1), ..., \phi(x_{n-1})))) =$
$\phi^{-1}(m_n(\phi(x_1), ..., \phi(x_{n-1}), m_{n-1}(\phi(x_1), ..., \phi(x_{n-1}))) =$
$\phi^{-1}(m_{n-1}(\phi(x_1), ..., \phi(x_{n-1}))) = m_{n-1}^\phi(x_1, ..., x_{n-1})$, which implies that
$(m_{n-1}^\phi, m_n^\phi) \in T_2$

For Type 3: Suppose that $(m_{n-1}, m_n) \in T_3$.
$m_{n-1}^\phi(m_n^\phi(x_1, x_1, x_2, ..., x_{n-1}), ..., m_n^\phi(x_1, x_2, ..., x_{n-1}, x_{n-1})) =$
$\phi^{-1}(m_{n-1}(\phi(m_n^\phi(x_1, x_1, x_2, ..., x_{n-1})), ..., \phi(m_n^\phi(x_1, x_2, ..., x_{n-1}, x_{n-1}))) =$
$\phi^{-1}(m_{n-1}(m_n(\phi(x_1), \phi(x_1), ..., \phi(x_{n-1})), ..., m_n(\phi(x_1), ..., \phi(x_{n-1}), \phi(x_{n-1}))))$
$= \phi^{-1}(m_{n-1}(\phi(x_1), ..., \phi(x_{n-1}))) = m_{n-1}^\phi(x_1, x_2, ..., x_{n-1})$, which implies that $(m_{n-1}^\phi, m_n^\phi) \in T_3$

Now it is trivial that the arithmetic mean $A_n(a_1,...,a_n) = \frac{\sum_{k=1}^{n} a_k}{n}$ is *type k invariant* with respect to the arithmetic mean $A_{n-1}(a_1,...,a_{n-1}) = \frac{\sum_{k=1}^{n-1} a_k}{n-1}$, $k =$ 1,2,3. Thus by Lemma 1, we have ■

Theorem 1 *If* $m_n(x_1,...,x_n) = h^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} h(x_k)\right)$ *and* $m_{n-1}(x_1,...,x_{n-1}) = h^{-1}\left(\frac{1}{n-1}\sum_{k=1}^{n-1} h(x_k)\right)$, *where* $h(x)$ *is continuous and monotone, on* $(0,\infty)$, *then* m_n *is type k invariant with respect to* $m_{n-1}, k = 1,2,3.$

Our next result is the converse of Theorem 1 for type 1 invariance among the class of weighted quasi arithmetic means for the same w and h.

Theorem 2 *Suppose that* $n \geq 3$ *and that* $m_n(x_1,...,x_n) = h^{-1}\left(\frac{\sum_{k=1}^{n} w(x_k)h(x_k)}{\sum_{k=1}^{n} w(x_k)}\right)$ *is type 1 invariant with respect to* $m_{n-1}(x_1,...,x_{n-1}) = h^{-1}\left(\frac{\sum_{k=1}^{n-1} w(x_k)h(x_k)}{\sum_{k=1}^{n-1} w(x_k)}\right)$, *where* $w(x)$ *is continuous and positive, and* $h(x)$ *is continuous and monotone, on* $(0,\infty)$. *Then* m_n *and* m_{n-1} *are each quasi–arithmetic means.*

Proof. By Lemma 1, we may assume that $h(x) = x$. Then (2.1), (2.3), and (2.4) of Proposition 1 become

$$\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) = \frac{n-1}{n^2}\frac{2w'}{w}, \tag{2.9}$$

$$\begin{aligned} \frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}) &= \frac{n-1}{n^3}\frac{3nww''-6(w')^2}{w^2} \\ \frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n]}) &= \frac{n-2}{(n-1)^3}\frac{3(n-1)ww''-6(w')^2}{w^2} \end{aligned} \tag{2.10}$$

and

$$\frac{\partial^3 m_n}{\partial x_1^2 \partial x_2}(x^{[n]}) = \frac{2(3-n)(w')^2-nww''}{n^3w^2}. \tag{2.11}$$

Substitute (2.9), (2.10), and (2.11) into (2.6) of Proposition 2 to obtain

$$\begin{aligned} &\frac{n(n^3-3n^2+3n-3)}{(n-1)^4}\frac{n-1}{n^3}\frac{3nww''-6(w')^2}{w^2} - \frac{3n}{(n-1)^3}\frac{2(3-n)(w')^2-nww''}{n^3w^2} \\ &-\frac{3n^3(n-2)}{(n-1)^5}\left(\frac{n-1}{n^2}\frac{2w'}{w}\right)^2 = \frac{n-2}{(n-1)^3}\frac{3(n-1)ww''-6(w')^2}{w^2}. \end{aligned} \tag{2.12}$$

Subtracting the right hand side of (2.12) and some simplification yields $\frac{3}{n}\frac{-2n(w'(x))^2+nw(x)w''(x)+4(w'(x))^2-2w(x)w''(x)}{(n-1)^3w^2(x)} = 0$, which implies that

$(n-2)w(x)w''(x)+(4-2n)(w'(x))^2=0 \Rightarrow w(x)w''(x)-2(w'(x))^2=0$ since $n>2$. If $w(x)$ is constant on $(0,\infty)$, then m_n and m_{n-1} are quasi–arithmetic means. Assuming then that $w(x)$ is not constant on $(0,\infty)$, we have $\frac{w''(x)}{w'(x)}=\frac{2w'(x)}{w(x)} \Rightarrow \frac{d}{dx}\log|w'(x)| = 2\frac{d}{dx}\log|w(x)| \Rightarrow \log|w'(x)| = \log\left(|w(x)|^2\right)+C_1 \Rightarrow w'(x)=C\left(w(x)\right)^2$ for some constant C. Solving for w yields $w(x)=-\frac{1}{Cx+D}$, where D is a constant, which implies that $w(x)=\frac{a}{x+b}$, where a and b are constants. Then $m_n(x_1,...,x_n)=\frac{\sum_{k=1}^{n}\frac{a_k}{a_k+b}}{\sum_{k=1}^{n}\frac{1}{a_k+b}}$ and $m_{n-1}(x_1,...,x_{n-1})=\frac{\sum_{k=1}^{n-1}\frac{a_k}{a_k+b}}{\sum_{k=1}^{n-1}\frac{1}{a_k+b}}$.

Let $g(x)=\dfrac{1}{x+b} \Rightarrow g^{-1}(x)=-b+\dfrac{1}{x}$ and $g^{-1}\left(\frac{1}{n}\sum_{k=1}^{n}g(a_k)\right)$

$$=-b+\frac{n}{\sum_{k=1}^{n}\frac{1}{a_k+b}}=\frac{-b\sum_{k=1}^{n}\frac{1}{a_k+b}+n}{\sum_{k=1}^{n}\frac{1}{a_k+b}}=\frac{-b\sum_{k=1}^{n}\frac{1}{a_k+b}+\sum_{k=1}^{n}\frac{a_k+b}{a_k+b}}{\sum_{k=1}^{n}\frac{1}{a_k+b}}=$$

$\frac{\sum_{k=1}^{n}\frac{a_k}{a_k+b}}{\sum_{k=1}^{n}\frac{1}{a_k+b}}=m_n(x_1,...,x_n)$. Similarly, $m_{n-1}(x_1,...,x_{n-1})=$

$g^{-1}\left(\frac{1}{n-1}\sum_{k=1}^{n-1}g(a_k)\right)$, which implies that m_n and m_{n-1} are quasi–arithmetic means. ■

Unlike the previous theorem, the following theorem shows that any two weighted quasi arithmetic means for the same w and h are type 2 invariant.

Theorem 3 $m_n(x_1,...,x_n)=h^{-1}\left(\frac{\sum_{k=1}^{n}w(x_k)h(x_k)}{\sum_{k=1}^{n}w(x_k)}\right)$ *is type 2 invariant with respect to* $m_{n-1}(x_1,...,x_{n-1})=h^{-1}\left(\frac{\sum_{k=1}^{n-1}w(x_k)h(x_k)}{\sum_{k=1}^{n-1}w(x_k)}\right)$ *for any fixed* h,w, *where* $w(x)$ *is continuous and positive, and* $h(x)$ *is continuous and monotone, on* $(0,\infty)$.

Proof. $m_n(x_1,...,x_{n-1},m_{n-1}(x_1,...,x_{n-1}))=m_{n-1}(x_1,...,x_{n-1}) \iff$
$h(m_n(x_1,...,x_{n-1},m_{n-1}(x_1,...,x_{n-1})))=h(m_{n-1}(x_1,...,x_{n-1})) \iff$
$\frac{\sum_{k=1}^{n-1}w(x_k)h(x_k)+w(m_{n-1}(x_1,...,x_{n-1}))h(m_{n-1}(x_1,...,x_{n-1}))}{\sum_{k=1}^{n-1}w(x_k)+w(m_{n-1}(x_1,...,x_{n-1}))}=$

$$\frac{\sum_{k=1}^{n-1} w(x_k)h(x_k)}{\sum_{k=1}^{n-1} w(x_k)} \iff \frac{\sum_{k=1}^{n-1} w(x_k)h(x_k)+w(m_{n-1}(x_1,...,x_{n-1}))\frac{\sum_{k=1}^{n-1} w(x_k)h(x_k)}{\sum_{k=1}^{n-1} w(x_k)}}{\sum_{k=1}^{n-1} w(x_k)+w(m_{n-1}(x_1,...,x_{n-1}))} =$$

$$\frac{\sum_{k=1}^{n-1} w(x_k)h(x_k)}{\sum_{k=1}^{n-1} w(x_k)} \iff \sum_{k=1}^{n-1} w(x_k) \sum_{k=1}^{n-1} w(x_k)h(x_k)+$$

$$w(m_{n-1}(x_1,...,x_{n-1})) \sum_{k=1}^{n-1} w(x_k)h(x_k) =$$

$$\sum_{k=1}^{n-1} w(x_k) \sum_{k=1}^{n-1} w(x_k)h(x_k) + w(m_{n-1}(x_1,...,x_{n-1})) \sum_{k=1}^{n-1} w(x_k)h(x_k) \quad \blacksquare$$

Our next result is the converse of Theorem 1 for type 3 invariance among the class of weighted quasi arithmetic means for the same w and h.

Theorem 4 *Suppose that* $n \geq 4$ *and that* $m_n(x_1, ..., x_n) = h^{-1}\left(\frac{\sum_{k=1}^{n} w(x_k)h(x_k)}{\sum_{k=1}^{n} w(x_k)}\right)$ *is type 3 invariant with respect to* $m_{n-1}(x_1, ..., x_{n-1}) = h^{-1}\left(\frac{\sum_{k=1}^{n-1} w(x_k)h(x_k)}{\sum_{k=1}^{n-1} w(x_k)}\right)$, *where* $w(x)$ *is continuous and positive, and* $h(x)$ *is continuous and monotone, on* $(0,\infty)$ *. Then* m_n *and* m_{n-1} *are each quasi–arithmetic means.*

Proof. As in the proof of Theorem 2, we may assume that $h(x) = x$. By (2.1) and (2.4) we have

$$\begin{aligned} \frac{\partial^3 m_{n-1}}{\partial x_1^2 \partial x_2}(x^{[n]}) &= \frac{2(4-n)(w')^2-(n-1)ww''}{(n-1)^3 w^2} \\ \frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n]}) &= \frac{n-2}{(n-1)^2}\frac{2w'}{w}. \end{aligned} \tag{2.13}$$

Substitute (2.9), (2.10), (2.11), and (2.13) into (2.8) of Proposition 4 to obtain

$$\begin{gathered} \frac{1}{n^2}\frac{3nww''-6(w')^2}{(w)^2} + \frac{6}{n-1}\frac{2(3-n)(w')^2-nww''}{n^3 w^2} + \\ \frac{(4-n^3)(n-2)}{n^3(n-1)^3}\frac{3(n-1)ww''-6(w')^2}{w^2} + \\ \frac{3(n-2)}{n^3}\frac{2(4-n)(w')^2-(n-1)ww''}{(n-1)^3 w^2} + \frac{12(n-3)(n-2)}{n^3(n-1)^2}\left(\frac{w'}{w}\right)^2 = 0. \end{gathered} \tag{2.14}$$

Some simplification of (2.14) yields $-3\frac{-2(w')^2 n^2+n^2 ww''-5nww''+10n(w')^2+6ww''-12(w')^2}{n^3(n-1)^2 w^2} = 0$, which implies that $(n^2-5n+6)w(x)w''(x)-2(n^2-5n+6)(w'(x))^2 = 0 \Rightarrow w(x)w''(x)-2(w'(x))^2 = 0$ if $n > 3$. The rest of the proof proceeds exactly as in the proof of Theorem 2. $\blacksquare$

Remark 5 *It is likely that Theorem 4 holds for $n = 3$ as well, but our method of proof becomes quite cumbersome in that case since one would have to look at partial derivatives of order 4. For example, if $m_2(x_1, x_2) = \frac{x_1^2+x_2^2}{x_1+x_2}$ and $m_3(x_1, x_2, x_3) = \frac{x_1^2+x_2^2+x_3^2}{x_1+x_2+x_3}$, then $\frac{\partial^k m_{n-1}}{\partial x_1^k}(x^{[n-1]}) = \frac{\partial^k m_n}{\partial x_1^k}(x^{[n]})$ for $k = 1$,2, and 3, but not $k = 4$. For $n \geq 4$, one only has $\frac{\partial^k m_{n-1}}{\partial x_1^k}(x^{[n-1]}) = \frac{\partial^k m_n}{\partial x_1^k}(x^{[n]})$ for $k = 1$ and 2 and for any weighted quasi–arithmetic means as in Theorem 4. Indeed, that is why our proof above of Theorem 4 works for $n \geq 4$.*

Remark 6 *It should be noted that equating second order partial derivatives in the proofs of Theorems 2 or 4 does not yield any information.*

3 Appendix

Proposition 3 is not used to prove any other results in this paper and we omit the proof. Before proving Propositions 2 and 4, we need the following result about symmetric functions, which we state without proof. For $x > 0$, we let $x^{[n]} = (x, ..., x) \in \Re_+^n$.

Lemma 2 *Let $E \subset R^n$ be an open region and let $f : E \to \Re$ be an n times differentiable, symmetric function. Assume that $D = \{(x_1, ..., x_n) \in E : x_1 = \cdots = x_n\}$ is nonempty. Let $i_1, ..., i_r$ be non–negative integers with $\sum_{i=1}^{r} i_j = v$, and let $\{i_{k_1}, ..., i_{k_r}\}$ be any permutation of $\{i_1, ..., i_r\}$. Then $\frac{\partial^v f}{\partial x_1^{i_1} \cdots \partial x_n^{i_r}}(x^{[n]}) = \frac{\partial^v f}{\partial x_1^{i_{k_1}} \cdots \partial x_n^{i_{k_r}}}(x^{[n]})$ for any $x > 0$.*

Now we need the following result about symmetric means.

Theorem 7 *Let m be a three times differentiable, symmetric mean in n variables, $n \geq 3$. Then for any $x > 0$,*
(i) $\frac{\partial m}{\partial x_k}(x^{[n]}) = \frac{1}{n}$
(ii) $\frac{\partial^2 m}{\partial x_i \partial x_j}(x^{[n]}) = -\frac{1}{n-1} \frac{\partial^2 m}{\partial x_i^2}(x^{[n]})$ *for* $i \neq j$
(iii) $\frac{\partial^3 m_n}{\partial x_1 \partial x_2 \partial x_3}(x^{[n]}) = -\frac{1}{(n-1)(n-2)} \frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}) - \frac{3}{n-2} \frac{\partial^3 m_n}{\partial x_1 \partial x_2^2}(x^{[n]})$ *for* i, j, k *distinct*

Proof. Take $\dfrac{d}{dx}$ of both sides of the identity $m(x^{[n]}) = x$. That gives

$$\sum_{k=1}^{n} \frac{\partial m}{\partial x_k}(x^{[n]}) = 1. \tag{3.1}$$

(i) then follows from Lemma 2. Taking $\dfrac{d}{dx}$ of both sides of (3.1) and using Clairaut's Theorem gives

$\sum_{k=1}^{n} \frac{\partial^2 m}{\partial x_1 \partial x_k}(x^{[n]}) + \cdots + \sum_{k=1}^{n} \frac{\partial^2 m}{\partial x_n \partial x_k}(x^{[n]}) = 0$, which implies that

$$\sum_{k=1}^{n} \frac{\partial^2 m}{\partial x_1^2}(x^{[n]}) + n(n-1)\frac{\partial^2 m}{\partial x_1 \partial x_2}(x^{[n]}) = 0. \tag{3.2}$$

by Lemma 2. (ii) then follows from Lemma 2. Taking $\dfrac{d}{dx}$ of both sides of (3.2) and using Clairaut's Theorem gives $\sum_{k=1}^{n} \frac{\partial^3 m}{\partial x_1 \partial x_1 \partial x_k}(x^{[n]})+$

$(n-1) \sum_{k=1}^{n} \frac{\partial^3 m}{\partial x_1 \partial x_2 \partial x_k}(x^{[n]}) = 0$, which implies that $\frac{\partial^3 m}{\partial x_1^3}(x^{[n]}) + \frac{\partial^3 m}{\partial x_1^2 \partial x_2}(x^{[n]}) +$ $\sum_{k=3}^{n} \frac{\partial^3 m}{\partial x_1^2 \partial x_k}(x^{[n]})+$

$(n-1)\frac{\partial^3 m}{\partial x_1^2 \partial x_2}(x^{[n]}) + (n-1)\frac{\partial^3 m}{\partial x_1 \partial x_2^2}(x^{[n]}) + (n-1)\sum_{k=3}^{n} \frac{\partial^3 m}{\partial x_1 \partial x_2 \partial x_k}(x^{[n]}) = 0.$

Lemma 2 again yields $\frac{\partial^3 m}{\partial x_1^3}(x^{[n]}) + 3(n-1)\frac{\partial^3 m}{\partial x_1^2 \partial x_2}(x^{[n]})+$

$(n-1)(n-2)\frac{\partial^3 m}{\partial x_1 \partial x_2 \partial x_3}(x^{[n]}) = 0$, which in turns gives (iii). ■

Remark 8 *Versions of Lemma 2 and Theorem 7 were given in [1] for* $n = 2, 3$.

Proof of Proposition 2

Proof. Let $\hat{x} = (x_1, ..., x_n)$. We find it convenient to use the following notation.

$$\pi_{\neq j}\hat{x} = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)$$
$$A = (m_{n-1}(\pi_{\neq 1}\hat{x}), m_{n-1}(\pi_{\neq 2}\hat{x}), ..., m_{n-1}(\pi_{\neq n}\hat{x})).$$

The notation $\pi_{\neq j}\hat{x}$ was introduced in [2]. By Definition 1,

$m_n(m_{n-1}(\pi_{\neq 1}\hat{x}), m_{n-1}(\pi_{\neq 2}\hat{x}), ..., m_{n-1}(\pi_{\neq n}\hat{x})) = m_n(\hat{x})$ for all $\hat{x} \in \Re^n$ and

$$m_n(A) = m_n(\hat{x}). \tag{3.3}$$

Since $m_{n-1}(x^{[n-1]}) = x$, for $x_1 = \cdots = x_n = x$, we have $A = x^{[n]}$. For any function of n variables, $g(x_1, ..., x_n)$,

$$\frac{\partial}{\partial x_k}(g(A)) = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j}(A)\frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_k} = \sum_{\substack{j=1 \\ j\neq k}}^{n} \frac{\partial g}{\partial x_j}(A)\frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_k}. \tag{3.4}$$

Thus, letting $g = m_n$ in (3.4), we have $\frac{\partial}{\partial x_k}(m_n(A)) = \sum_{j=1}^{n} \frac{\partial m_n}{\partial x_j}(A)\frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_k}$, which implies that

$$\frac{\partial}{\partial x_k}(m_n(A)) = \sum_{\substack{j=1 \\ j\neq k}}^{n} \frac{\partial m_n}{\partial x_j}(A)\frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_k}. \tag{3.5}$$

Note that $\frac{\partial m_n}{\partial x_j}(A)$ means $\frac{\partial m_n}{\partial x_j}$ evaluated at A, which is not the same as

$\frac{\partial}{\partial x_k}\left(m_n(A)\right)=\frac{\partial}{\partial x_k}m_n(m_{n-1}(x_2,...,x_n),...,m_{n-1}(x_1,...,x_{n-1}))$.

Letting $g=\frac{\partial m_n}{\partial x_j}$ in (3.4), we have $\frac{\partial}{\partial x_k}\left(\frac{\partial m_n}{\partial x_j}(A)\right)=\sum_{i=1}^{n}\frac{\partial^2 m_n}{\partial x_i\partial x_j}(A)\frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_k}=\sum_{\substack{i=1\\i\neq k}}^{n}\frac{\partial^2 m_n}{\partial x_i\partial x_j}(A)\frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_k}$. Letting $k=1$ yields

$$\frac{\partial}{\partial x_1}\left(\frac{\partial m_n}{\partial x_j}(A)\right)=\sum_{i=2}^{n}\frac{\partial^2 m_n}{\partial x_i\partial x_j}(A)\frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1}. \tag{3.6}$$

Using (3.5) & (3.6), $\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_k}\left(m_n(A)\right)\right)=\frac{\partial}{\partial x_1}\left(\sum_{\substack{j=1\\j\neq k}}^{n}\frac{\partial m_n}{\partial x_j}(A)\frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_k}\right)=\sum_{\substack{j=1\\j\neq k}}^{n}\left(\frac{\partial m_n}{\partial x_j}(A)\frac{\partial^2 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1\partial x_k}+\frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_k}\frac{\partial}{\partial x_1}\left(\frac{\partial m_n}{\partial x_j}(A)\right)\right)$, which implies that

$$\frac{\partial^2}{\partial x_1\partial x_k}\left(m_n(A)\right)= \tag{3.7}$$
$$\sum_{\substack{j=1\\j\neq k}}^{n}\left(\frac{\partial m_n}{\partial x_j}(A)\frac{\partial^2 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1\partial x_k}+\frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_k}\sum_{i=2}^{n}\frac{\partial^2 m_n}{\partial x_i\partial x_j}(A)\frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1}\right).$$

Let $k=1$ in (3.7) to obtain

$$\frac{\partial^2}{\partial x_1^2}\left(m_n(A)\right)= \tag{3.8}$$
$$\sum_{j=2}^{n}\left(\frac{\partial m_n}{\partial x_j}(A)\frac{\partial^2 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^2}+\frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1}\sum_{i=2}^{n}\frac{\partial^2 m_n}{\partial x_i\partial x_j}(A)\frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1}\right).$$

(3.8) and $\frac{\partial^2}{\partial x_1^2}\left(m_n(A)\right)=\frac{\partial^2}{\partial x_1^2}\left(m_n(\hat{x})\right)$ imply that

$$\sum_{j=2}^{n}\left(\frac{\partial m_n}{\partial x_j}(A)\frac{\partial^2 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^2}+\frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1}\sum_{i=2}^{n}\frac{\partial^2 m_n}{\partial x_i\partial x_j}(A)\frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1}\right)=\frac{\partial^2 m_n}{\partial x_1^2}(\hat{x}). \tag{3.9}$$

Now let $x_1=\cdots=x_n=x$ in (3.9):

$\sum_{j=2}^{n}\left(\frac{\partial m_n}{\partial x_j}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\frac{\partial m_{n-1}}{\partial x_1}(x^{[n-1]})\sum_{i=2}^{n}\frac{\partial^2 m_n}{\partial x_i\partial x_j}(x^{[n]})\frac{\partial m_{n-1}}{\partial x_1}(x^{[n-1]})\right)$
$=\frac{\partial^2 m_n}{\partial x_1^2}\left(x^{[n]}\right)\Rightarrow$

$$\sum_{j=2}^{n}\left(\frac{1}{n}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\frac{1}{(n-1)^2}\sum_{\substack{i=2\\i\neq j}}^{n}\frac{\partial^2 m_n}{\partial x_i\partial x_j}(x^{[n]})+\frac{1}{(n-1)^2}\frac{\partial^2 m_n}{\partial x_j^2}(x^{[n]})\right)= \frac{\partial^2 m_n}{\partial x_1^2}\left(x^{[n]}\right) \tag{3.10}$$

since $\frac{\partial m_{n-1}}{\partial x_1}(x^{[n-1]})=\frac{1}{n-1}$, $\frac{\partial m_n}{\partial x_j}(x^{[n]})=\frac{1}{n}$ by Theorem 7(i). Note that if $x_1=\cdots=x_n=x$ and $j\geq 2$, then $\left[\frac{\partial^k m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^k}\right]_{x_1=\cdots=x_{n-1}=x}=\frac{\partial^k m_{n-1}}{\partial x_1^k}(x^{[n-1]})$ for

$k = 1, 2, 3$. It also follows from Lemma 2 that $\frac{\partial^2 m_n}{\partial x_j^2} = \frac{\partial^2 m_n}{\partial x_1^2}$ and $\frac{\partial^2 m_n}{\partial x_i \partial x_j}(x^{[n]}) = \frac{\partial^2 m_n}{\partial x_1 \partial x_2}(x^{[n]})$ for any $j \geq 2$. Thus (3.10) becomes

$$\frac{n-1}{n} \frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) + \frac{n-2}{n-1} \frac{\partial^2 m_n}{\partial x_1 \partial x_2}(x^{[n]}) + \frac{1}{n-1} \frac{\partial^2 m_n}{\partial x_1^2}\left(x^{[n]}\right) = \frac{\partial^2 m_n}{\partial x_1^2}\left(x^{[n]}\right),$$

which implies that $\frac{n-1}{n} \frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) = \frac{n(n-2)}{(n-1)^2} \frac{\partial^2 m_n}{\partial x_1^2}\left(x^{[n]}\right)$, which in turn gives (2.5). Now we derive an expression involving third order partials. Letting $g = \frac{\partial^2 m_n}{\partial x_i \partial x_j}$ in (3.4) yields $\frac{\partial}{\partial x_k}\left(\frac{\partial^2 m_n}{\partial x_i \partial x_j}(A)\right) = \sum_{\substack{l=1 \\ l \neq k}}^{n} \frac{\partial\left(\frac{\partial^2 m_n}{\partial x_i \partial x_j}\right)}{\partial x_l}(A) \frac{\partial m_{n-1}(\pi_{\neq l}\hat{x})}{\partial x_k}$, which implies that

$$\frac{\partial}{\partial x_k}\left(\frac{\partial^2 m_n}{\partial x_i \partial x_j}(A)\right) = \sum_{\substack{l=1 \\ l \neq k}}^{n} \frac{\partial^3 m_n}{\partial x_i \partial x_j \partial x_l}(A) \frac{\partial m_{n-1}(\pi_{\neq l}\hat{x})}{\partial x_k}. \tag{3.11}$$

Using (3.8),

$$\frac{\partial}{\partial x_1}\left(\frac{\partial^2}{\partial x_1^2}\left(m_n(A)\right)\right) =$$

$$\frac{\partial}{\partial x_1} \sum_{j=2}^{n} \left(\frac{\partial m_n}{\partial x_j}(A) \frac{\partial^2 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^2} + \frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1} \sum_{i=2}^{n} \frac{\partial^2 m_n}{\partial x_i \partial x_j}(A) \frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1}\right) =$$

$$\sum_{j=2}^{n} \left(\frac{\partial}{\partial x_1}\left(\frac{\partial m_n}{\partial x_j}(A) \frac{\partial^2 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^2}\right) + \frac{\partial}{\partial x_1}\left(\frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1} \sum_{i=2}^{n} \frac{\partial^2 m_n}{\partial x_i \partial x_j}(A) \frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1}\right)\right) =$$

$$\sum_{j=2}^{n} \Bigg(\frac{\partial m_n}{\partial x_j}(A) \frac{\partial^3 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^3} + \frac{\partial^2 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^2} \frac{\partial}{\partial x_1}\left(\frac{\partial m_n}{\partial x_j}(A)\right) + \frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1} \sum_{i=2}^{n} \left(\frac{\partial^2 m_n}{\partial x_i \partial x_j}(A) \frac{\partial^2 m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1^2} + \frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1} \frac{\partial}{\partial x_1}\left(\frac{\partial^2 m_n}{\partial x_i \partial x_j}(A)\right)\right) + \frac{\partial^2 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^2} \sum_{i=2}^{n} \frac{\partial^2 m_n}{\partial x_i \partial x_j}(A) \frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1}\Bigg).$$

Hence

$$\frac{\partial^3}{\partial x_1^3}\left(m_n(A)\right) = \sum_{j=2}^{n} \Bigg(\frac{\partial m_n}{\partial x_j}(A) \frac{\partial^3 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^3} + \frac{\partial^2 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^2} \frac{\partial}{\partial x_1}\left(\frac{\partial m_n}{\partial x_j}(A)\right) + \sum_{i=2}^{n} \Bigg(\frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1} \left(\frac{\partial^2 m_n}{\partial x_i \partial x_j}(A) \frac{\partial^2 m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1^2} + \frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1} \frac{\partial}{\partial x_1}\left(\frac{\partial^2 m_n}{\partial x_i \partial x_j}(A)\right)\right) + \frac{\partial^2 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^2} \frac{\partial^2 m_n}{\partial x_i \partial x_j}(A) \frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1}\Bigg)\Bigg). \tag{3.12}$$

Consider first the first line of (3.12). By (3.6),

$$\sum_{j=2}^{n}\left(\frac{\partial m_n}{\partial x_j}(A)\frac{\partial^3 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^3}+\frac{\partial^2 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^2}\frac{\partial}{\partial x_1}\left(\frac{\partial m_n}{\partial x_j}(A)\right)\right)=$$

$$\sum_{j=2}^{n}\left(\frac{\partial m_n}{\partial x_j}(A)\frac{\partial^3 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^3}+\frac{\partial^2 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^2}\sum_{l=2}^{n}\frac{\partial^2 m_n}{\partial x_l\partial x_j}(A)\frac{\partial m_{n-1}(\pi_{\neq l}\hat{x})}{\partial x_1}\right). \quad (3.13)$$

Let $x_1=\cdots=x_n=x$ in (3.13):

$$\sum_{j=2}^{n}\Big(\frac{\partial m_n}{\partial x_j}(x^{[n]})\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]})+$$
$$\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\sum_{l=2}^{n}\frac{\partial^2 m_n}{\partial x_l\partial x_j}(x^{[n]})\frac{\partial m_{n-1}}{\partial x_1}(x^{[n-1]})\Big)=$$
$$\sum_{j=2}^{n}\left(\frac{1}{n}\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]})+\frac{1}{n-1}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\sum_{l=2}^{n}\frac{\partial^2 m_n}{\partial x_l\partial x_j}(x^{[n]})\right)$$

since $\frac{\partial m_{n-1}}{\partial x_1}(x^{[n-1]})=\frac{1}{n-1}, \frac{\partial m_n}{\partial x_j}(x^{[n]})=\frac{1}{n}$ by Theorem 7(i). Breaking up the summation over i yields

$$\sum_{j=2}^{n}\Big(\frac{1}{n}\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]})+$$
$$\frac{1}{n-1}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\sum_{\substack{l=2\\ l\neq j}}^{n}\frac{\partial^2 m_n}{\partial x_l\partial x_j}(x^{[n]})+\frac{1}{n-1}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\frac{\partial^2 m_n}{\partial x_j^2}(x^{[n]})\Big).$$

By Lemma 2, $\frac{\partial^2 m_n}{\partial x_l\partial x_j}(x^{[n]})=\frac{\partial^2 m_n}{\partial x_1\partial x_2}(x^{[n]})$ for $i\neq j$ and $\frac{\partial^2 m_n}{\partial x_j^2}(x^{[n]})=\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})$, which gives

$$\frac{n-1}{n}\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]})+(n-2)\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\frac{\partial^2 m_n}{\partial x_1\partial x_2}(x^{[n]})+ \quad (3.14)$$
$$\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})$$

for the first line of (3.12) with $x_1=\cdots=x_n=x$. For the second and third lines of (3.12), by (3.11), we have

$$\sum_{j=2}^{n}\Big(\frac{\partial m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1}\sum_{i=2}^{n}\frac{\partial^2 m_n}{\partial x_i\partial x_j}(A)\frac{\partial^2 m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1^2}+$$
$$\sum_{i=2}^{n}\Big(\frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1}\sum_{l=2}^{n}\frac{\partial^3 m_n}{\partial x_i\partial x_j\partial x_l}(A)\frac{\partial m_{n-1}(\pi_{\neq l}\hat{x})}{\partial x_1}+ \quad (3.15)$$
$$\frac{\partial^2 m_{n-1}(\pi_{\neq j}\hat{x})}{\partial x_1^2}\frac{\partial^2 m_n}{\partial x_i\partial x_j}(A)\frac{\partial m_{n-1}(\pi_{\neq i}\hat{x})}{\partial x_1}\Big)\Big).$$

Let $x_1 = \cdots = x_n = x$ in (3.15):

$$\frac{1}{n-1}\sum_{j=2}^{n}\Big(\sum_{i=2}^{n}\Big(\frac{\partial^2 m_n}{\partial x_i\partial x_j}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\frac{1}{(n-1)^2}\sum_{l=2}^{n}\frac{\partial^3 m_n}{\partial x_i\partial x_j\partial x_l}(x^{[n]})+ \tag{3.16}$$
$$\frac{1}{n-1}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\frac{\partial^2 m_n}{\partial x_i\partial x_j}(x^{[n]})\Big)\Big),$$

which represents the second and third lines of (3.12) with $x_1 = \cdots = x_n = x$. Consider first the first line of (3.16), **without** the $\frac{1}{n-1}$ factor. We break up the summation over i as follows:

$$\sum_{j=2}^{n}\Big(\sum_{i=2}^{n}\Big(\frac{\partial^2 m_n}{\partial x_i\partial x_j}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\frac{1}{(n-1)^2}\sum_{l=2}^{n}\frac{\partial^3 m_n}{\partial x_i\partial x_j\partial x_l}(x^{[n]})\Big)\Big)=$$
$$\sum_{j=2}^{n}\Big(\sum_{\substack{i=2\\ i\neq j}}^{n}\Big(\frac{\partial^2 m_n}{\partial x_i\partial x_j}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\frac{1}{(n-1)^2}\sum_{l=2}^{n}\frac{\partial^3 m_n}{\partial x_i\partial x_j\partial x_l}(x^{[n]})\Big)+$$
$$\frac{\partial^2 m_n}{\partial x_j^2}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\frac{1}{(n-1)^2}\sum_{l=2}^{n}\frac{\partial^3 m_n}{\partial x_j^2\partial x_l}(x^{[n]})\Big).$$

Now break up $\sum_{l=2}^{n}\frac{\partial^3 m_n}{\partial x_i\partial x_j\partial x_l}(x^{[n]})$ into three summations, one with $l = i, l \neq j$, $l \neq i, l - j$, and $l \neq i, l \neq j$, and break up $\sum_{l=2}^{n}\frac{\partial^3 m_n}{\partial x_j^2\partial x_l}(x^{[n]})$ into two summations, one with $l = j$ and one with $l \neq j$, which yields

$$\sum_{j=2}^{n}\Big(\sum_{\substack{i=2\\ i\neq j}}^{n}\Big(\frac{\partial^2 m_n}{\partial x_i\partial x_j}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\Big)+$$
$$\frac{1}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_i\partial x_j^2}(x^{[n]})+\frac{1}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_i^2\partial x_j}(x^{[n]})+\frac{1}{(n-1)^2}\sum_{\substack{l=2\\ l\neq i, l\neq j}}^{n}\frac{\partial^3 m_n}{\partial x_i\partial x_j\partial x_l}(x^{[n]})\Big)+$$
$$\frac{\partial^2 m_n}{\partial x_j^2}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\frac{1}{(n-1)^2}\sum_{\substack{l=2\\ l\neq j}}^{n}\frac{\partial^3 m_n}{\partial x_j^2\partial x_l}(x^{[n]})+\frac{1}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_j^3}(x^{[n]})\Big)\Big).$$

Lemma 2 implies that $\frac{\partial^3 m_n}{\partial x_i\partial x_j^2}(x^{[n]}) = \frac{\partial^3 m_n}{\partial x_j^2\partial x_l}(x^{[n]}) = \frac{\partial^3 m_n}{\partial x_1^2\partial x_2}(x^{[n]})$, which gives

$$\sum_{j=2}^{n}\Big(\sum_{\substack{i=2\\ i\neq j}}^{n}\frac{\partial^2 m_n}{\partial x_1\partial x_2}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\frac{1}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_1^2\partial x_2}(x^{[n]})+$$
$$\frac{1}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_1^2\partial x_2}(x^{[n]})+\frac{n-3}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_1\partial x_2\partial x_3}(x^{[n]})+$$
$$\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\frac{n-2}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_1^2\partial x_2}(x^{[n]})+\frac{1}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]})\Big)=$$

$$\sum_{j=2}^{n}\sum_{\substack{i=2\\i\neq j}}^{n}(\left[\tfrac{\partial^2 m_n}{\partial x_1\partial x_2}(x^{[n]})\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\right]+$$

$$\tfrac{2}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1\partial x_2^2}(x^{[n]})+\tfrac{n-3}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1\partial x_2\partial x_3}(x^{[n]}))+$$

$$(n-1)\left(\tfrac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\tfrac{n-2}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1^2\partial x_2}(x^{[n]})+\tfrac{1}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1^3}(x^{[n]})\right)=$$

$$\begin{aligned} = \quad & (n-1)(n-2)\times(\tfrac{\partial^2 m_n}{\partial x_1\partial x_2}(x^{[n]})\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\\ & \tfrac{2}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1\partial x_2^2}(x^{[n]})+\tfrac{n-3}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1\partial x_2\partial x_3}(x^{[n]}))+\\ & (n-1)\times\\ & \left(\tfrac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\tfrac{n-2}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1^2\partial x_2}(x^{[n]})+\tfrac{1}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1^3}(x^{[n]})\right).\end{aligned}$$

Now multiply thru by $\frac{1}{n-1}$ and distribute the $n-2$ to obtain the first line of (3.16) **with** the $\frac{1}{n-1}$ factor:

$$\begin{gathered}(n-2)\tfrac{\partial^2 m_n}{\partial x_1\partial x_2}(x^{[n]})\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\tfrac{3(n-2)}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1\partial x_2^2}(x^{[n]})+\\ \tfrac{(n-2)(n-3)}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1\partial x_2\partial x_3}(x^{[n]})+\tfrac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\\ \tfrac{1}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}).\end{gathered} \tag{3.17}$$

For the second line of (3.16) **with** the $\frac{1}{n-1}$ factor, we break up the summation over i as follows:

$$\tfrac{1}{n-1}\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\sum_{j=2}^{n}(\sum_{\substack{i=2\\i\neq j}}^{n}\tfrac{\partial^2 m_n}{\partial x_i\partial x_j}(x^{[n]})+\tfrac{\partial^2 m_n}{\partial x_j^2}(x^{[n]})))=$$

$$\tfrac{1}{n-1}\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\sum_{j=2}^{n}((n-2)\tfrac{\partial^2 m_n}{\partial x_1\partial x_2}(x^{[n]})+\tfrac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}))$$

using Lemma 2, which gives

$$(n-2)\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\tfrac{\partial^2 m_n}{\partial x_1\partial x_2}(x^{[n]})+\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\tfrac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}). \tag{3.18}$$

Add (3.17) and (3.18) to obtain

$$\begin{gathered}(2n-4)\tfrac{\partial^2 m_n}{\partial x_1\partial x_2}(x^{[n]})\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\tfrac{3(n-2)}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1\partial x_2^2}(x^{[n]})+ \\ \tfrac{(n-2)(n-3)}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1\partial x_2\partial x_3}(x^{[n]})+2\tfrac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\tfrac{1}{(n-1)^2}\tfrac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}).\end{gathered} \tag{3.19}$$

(3.19) equals (3.16) with $x_1 = \cdots = x_n = x$. Now add (3.14) and (3.19), which yields (3.12) with $x_1 = \cdots = x_n = x$:

$$\begin{gathered}(3n-6)\frac{\partial^2 m_n}{\partial x_1 \partial x_2}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) + \frac{3(n-2)}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_1 \partial x_2^2}(x^{[n]}) + \\ \frac{(n-2)(n-3)}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_1 \partial x_2 \partial x_3}(x^{[n]}) + 3\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) + \\ \frac{1}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}) + \frac{n-1}{n}\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]}).\end{gathered} \tag{3.20}$$

Substitute into (3.20), using $\frac{\partial^2 m_n}{\partial x_1 \partial x_2}(x^{[n]}) = -\frac{1}{n-1}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})$ and $\frac{\partial^3 m_n}{\partial x_1 \partial x_2 \partial x_3}(x^{[n]}) = -\frac{1}{(n-1)(n-2)}\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}) - \frac{3}{n-2}\frac{\partial^3 m_n}{\partial x_1 \partial x_2^2}(x^{[n]})$ from heorem 7. That yields

$$\begin{gathered}-\frac{3n-6}{n-1}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) + \frac{3(n-2)}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_1 \partial x_2^2}(x^{[n]}) + \\ \frac{(n-2)(n-3)}{(n-1)^2}\left(-\frac{1}{(n-1)(n-2)}\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}) - \frac{3}{n-2}\frac{\partial^3 m_n}{\partial x_1 \partial x_2^2}(x^{[n]})\right) + \\ 3\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) + \frac{1}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}) + \frac{n-1}{n}\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]}).\end{gathered}$$

Thus

$$\begin{aligned}\frac{\partial^3}{\partial x_1^3}(m_n(A)) &= \frac{3}{n-1}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) + \frac{3}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_1 \partial x_2^2}(x^{[n]}) + \\ &\quad \frac{2}{(n-1)^3}\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}) + \frac{n-1}{n}\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]}).\end{aligned}$$

$\frac{\partial^3}{\partial x_1^3}(m_n(A)) = \frac{\partial^3}{\partial x_1^3}m_n(\hat{x}) \Rightarrow \frac{\partial^3}{\partial x_1^3}(m_n(A))(x^{[n]}) = \frac{\partial^3}{\partial x_1^3}m_n(\hat{x})(x^{[n]}) \Rightarrow$

$$\begin{gathered}\frac{3}{n-1}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) + \frac{3}{(n-1)^2}\frac{\partial^3 m_n}{\partial x_1 \partial x_2^2}(x^{[n]}) - \\ \frac{n^3-3n^2+3n-3}{(n-1)^3}\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}) + \frac{n-1}{n}\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]}) = 0.\end{gathered} \tag{3.21}$$

One can also use (2.5) to substitute for $\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})$ in (3.21). That gives (2.6). ■

Proof of Proposition 4

Proof. Let $\hat{x} = (x_1, ..., x_{n-1}) \in \Re_+^{n-1}$. We find it convenient to use the following notation:

$$\begin{aligned}\pi_{=j}\hat{x} &= (x_1, ..., x_{j-1}, x_j, x_j, x_{j+1}, ..., x_{n-1}) \in \Re_+^n, \\ A &= (m_n(\pi_{=1}\hat{x}), ..., m_n(\pi_{=n-1}\hat{x})) \in \Re_+^{n-1}.\end{aligned}$$

Thus m_n is *type 3 invariant* with respect to m_{n-1} if

$$m_{n-1}(A) = m_{n-1}(\hat{x}). \tag{3.22}$$

For $x_1 = \cdots = x_{n-1} = x$, we have $A = x^{[n-1]} = (x, ..., x) \in \Re_+^{n-1}$. For any function of $n-1$ variables, $g(x_1, ..., x_{n-1})$, $\frac{\partial}{\partial x_k}(g(A)) =$

$\frac{\partial}{\partial x_k} g(m_n(\pi_{=1}\hat{x}), ..., m_n(\pi_{=n-1}\hat{x})) \Rightarrow \frac{\partial}{\partial x_k}\left(g(A)\right) = \sum_{j=1}^{n-1} \frac{\partial g}{\partial x_j}(A) \frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_k}$,

which implies that

$$\frac{\partial}{\partial x_1}\left(g(A)\right) = \sum_{j=1}^{n-1} \frac{\partial g}{\partial x_j}(A) \frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}. \tag{3.23}$$

Thus, letting $g = m_{n-1}$ in (3.23), we have

$$\frac{\partial}{\partial x_1}(m_{n-1}(A)) = \sum_{j=1}^{n-1} \frac{\partial m_{n-1}}{\partial x_j}(A) \frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}. \tag{3.24}$$

Take $\frac{\partial}{\partial x_1}$ of both sides of (3.22): $\frac{\partial}{\partial x_1}(m_{n-1}(A)) = \frac{\partial}{\partial x_1}(m_{n-1}(\hat{x})) \Rightarrow$

$$\sum_{j=1}^{n-1} \frac{\partial m_{n-1}}{\partial x_j}(A) \frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1} = \frac{\partial m_{n-1}}{\partial x_1}(\hat{x}). \tag{3.25}$$

Letting $g = \frac{\partial m_{n-1}}{\partial x_j}$ in (3.23) and replacing the index of summation, j, by i, we have

$$\frac{\partial}{\partial x_1}\left(\frac{\partial m_{n-1}}{\partial x_j}(A)\right) = \sum_{i=1}^{n-1} \frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A) \frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}. \tag{3.26}$$

Taking $\frac{\partial}{\partial x_1}$ of both sides of (3.25) gives $\frac{\partial}{\partial x_1}\left(\sum_{j=1}^{n-1} \frac{\partial m_{n-1}}{\partial x_j}(A) \frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\right) =$ $\frac{\partial^2 m_{n-1}}{\partial x_1^2}(\hat{x}) \Rightarrow \sum_{j=1}^{n-1}\left[\frac{\partial m_{n-1}}{\partial x_j}(A) \frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2} + \frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1} \frac{\partial}{\partial x_1}\left(\frac{\partial m_{n-1}}{\partial x_j}(A)\right)\right] =$ $\frac{\partial^2 m_{n-1}}{\partial x_1^2}(\hat{x})$, which by (3.26) gives

$$\sum_{j=1}^{n-1}\left[\frac{\partial m_{n-1}}{\partial x_j}(A) \frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2} + \frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1} \sum_{i=1}^{n-1} \frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A) \frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}\right] = \frac{\partial^2 m_{n-1}}{\partial x_1^2}(\hat{x}). \tag{3.27}$$

Before going further, we need a formula for $\frac{\partial^k m_n(\pi_{=j}\hat{x})}{\partial x_1^k}$ on the diagonal. First, it follows immediately that, for $k = 1,2,3$

$$\left[\frac{\partial^k m_n(\pi_{=j}\hat{x})}{\partial x_1^k}\right]_{x_1=\cdots=x_{n-1}=x} = \left[\frac{\partial^k m_n}{\partial x_1^k}\right]_{x_1=\cdots=x_n=x} = \frac{\partial^k m_n}{\partial x_1^k}(x^{[n]}), j \geq 2. \tag{3.28}$$

Also,

$$\frac{\partial}{\partial x}(m_n\left(x, x, x_2, ..., x_{n-1}\right)) = \frac{\partial m_n}{\partial x_1}\left(x, x, x_2, ..., x_{n-1}\right) + \frac{\partial m_n}{\partial x_2}\left(x, x, x_2, ..., x_{n-1}\right),$$

which implies that

$$\left[\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}\right]_{x_1=\cdots=x_{n-1}=x} = \frac{2}{n}. \tag{3.29}$$

$$\frac{\partial^2}{\partial x^2}(m_n(x,x,x_2,...,x_{n-1}) = \frac{\partial^2 m_n}{\partial x_1^2}(x,x,x_2,...,x_{n-1}) + \frac{\partial^2 m_n}{\partial x_1 \partial x_2}(x,x,x_2,...,x_{n-1}) + \frac{\partial^2 m_n}{\partial x_2 \partial x_1}(x,x,x_2,...,x_{n-1}) + \frac{\partial^2 m_n}{\partial x_2^2}(x,x,x_2,...,x_{n-1}).$$

Thus $\frac{\partial^2}{\partial x^2}(m_n(x,x,x_2,...,x_{n-1})) = \frac{\partial^2 m_n}{\partial x_1^2}(x,x,x_2,...,x_{n-1}) + 2\frac{\partial^2 m_n}{\partial x_1 \partial x_2}(x,x,x_2,...,x_{n-1}) + \frac{\partial^2 m_n}{\partial x_2^2}(x,x,x_2,...,x_{n-1})$. Letting $x_1 = \cdots = x_{n-1} = x$ yields $2\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) + 2\left(-\frac{1}{n-1}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\right)$, which implies that

$$\left[\frac{\partial^2 m_n(\pi_{=1}\hat{x})}{\partial x_1^2}\right]_{x_1=\cdots=x_{n-1}=x} = \frac{2(n-2)}{n-1}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}). \tag{3.30}$$

$$\frac{\partial^3}{\partial x^3}(m_n(x,x,x_2,...,x_{n-1})) = \frac{\partial^3 m_n}{\partial x_1^3}(x,x,x_2,...,x_{n-1}) + \frac{\partial^3 m_n}{\partial x_1^2 \partial x_2}(x,x,x_2,...,x_{n-1}) + 2\frac{\partial^3 m_n}{\partial x_1^2 \partial x_2}(x,x,x_2,...,x_{n-1}) + 2\frac{\partial^3 m_n}{\partial x_1 \partial x_2^2}(x,x,x_2,...,x_{n-1}) + \frac{\partial^3 m_n}{\partial x_1 \partial x_2^2}(x,x,x_2,...,x_{n-1}) + \frac{\partial^3 m_n}{\partial x_2^3}(x,x,x_2,...,x_{n-1}).$$

Letting $x_1 = \cdots = x_{n-1} = x$ implies that

$$\left[\frac{\partial^3 m_n(\pi_{=1}\hat{x})}{\partial x_1^3}\right]_{x_1=\cdots=x_{n-1}=x} = 2\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]}) + 6\frac{\partial^3 m_n}{\partial x_1^2 \partial x_2}(x^{[n]}). \tag{3.31}$$

Rewrite (3.27) as

$$\frac{\partial m_{n-1}}{\partial x_1}(A)\frac{\partial^2 m_n(\pi_{=1}\hat{x})}{\partial x_1^2} + \frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}\sum_{i=1}^{n-1}\frac{\partial^2 m_{n-1}}{\partial x_1 \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1} + \sum_{j=2}^{n-1}\left[\frac{\partial m_{n-1}}{\partial x_j}(A)\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2} + \frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\sum_{i=1}^{n-1}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}\right] = \frac{\partial^2 m_{n-1}}{\partial x_1^2}(\hat{x}),$$

which implies that

$$\frac{\partial m_{n-1}}{\partial x_1}(A)\frac{\partial^2 m_n(\pi_{=1}\hat{x})}{\partial x_1^2} + \frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(A)\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1} + \frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}\sum_{i=2}^{n-1}\frac{\partial^2 m_{n-1}}{\partial x_1 \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1} + \sum_{j=2}^{n-1}\left[\frac{\partial m_{n-1}}{\partial x_j}(A)\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2} + \frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\sum_{i=1}^{n-1}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}\right] = \frac{\partial^2 m_{n-1}}{\partial x_1^2}(\hat{x}).$$

Hence

$$
\begin{gathered}
\frac{\partial m_{n-1}}{\partial x_1}(A)\frac{\partial^2 m_n(\pi_{=1}\hat{x})}{\partial x_1^2} + \left(\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}\right)^2 \frac{\partial^2 m_{n-1}}{\partial x_1^2}(A) + \\
\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}\sum_{i=2}^{n-1}\frac{\partial^2 m_{n-1}}{\partial x_1 \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1} + \\
\sum_{j=2}^{n-1}\Big(\frac{\partial m_{n-1}}{\partial x_j}(A)\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2} + \frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_1}(A)\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1} + \\
\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\sum_{\substack{i=2\\ i\neq j}}^{n-1}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1} + \frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\frac{\partial^2 m_{n-1}}{\partial x_j^2}(A)\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\Big) = \\
\frac{\partial^2 m_{n-1}}{\partial x_1^2}(\hat{x}),
\end{gathered}
$$

which implies that

$$
\begin{gathered}
\frac{\partial m_{n-1}}{\partial x_1}(A)\frac{\partial^2 m_n(\pi_{=1}\hat{x})}{\partial x_1^2} + \left(\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}\right)^2 \frac{\partial^2 m_{n-1}}{\partial x_1^2}(A) + \\
\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}\sum_{i=2}^{n-1}\frac{\partial^2 m_{n-1}}{\partial x_1 \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1} + \\
\sum_{j=2}^{n-1}\Big(\frac{\partial m_{n-1}}{\partial x_j}(A)\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2} + \frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_1}(A)\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1} + \qquad (3.32) \\
\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\sum_{\substack{i=2\\ i\neq j}}^{n-1}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1} + \left(\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\right)^2\frac{\partial^2 m_{n-1}}{\partial x_j^2}(A)\Big) = \\
\frac{\partial^2 m_{n-1}}{\partial x_1^2}(\hat{x}).
\end{gathered}
$$

Now let $x_1 = \cdots = x_{n-1} = x$ in (3.32) and use (3.28)–(3.30), Lemma 2, and Theorem 7:

$$
\begin{gathered}
\tfrac{1}{n-1}\tfrac{2(n-2)}{n-1}\tfrac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) + \tfrac{4}{n^2}\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) + \\
\tfrac{2}{n^2}\sum_{i=2}^{n-1}\tfrac{\partial^2 m_{n-1}}{\partial x_1 \partial x_i}(x^{[n-1]}) + \sum_{j=2}^{n-1}\Big(\tfrac{1}{n-1}\tfrac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) + \tfrac{2}{n^2}\tfrac{\partial^2 m_{n-1}}{\partial x_j \partial x_1}(x^{[n-1]}) + \\
\tfrac{1}{n^2}\sum_{\substack{i=2\\ i\neq j}}^{n-1}\tfrac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(x^{[n-1]}) + \tfrac{1}{n^2}\tfrac{\partial^2 m_{n-1}}{\partial x_j^2}(x^{[n-1]})\Big) = \\
\tfrac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}).
\end{gathered}
$$

Thus

$$\frac{2(n-2)}{(n-1)^2}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) + \frac{4}{n^2}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) +$$
$$\frac{2(n-2)}{n^2}\frac{\partial^2 m_{n-1}}{\partial x_1 \partial x_2}(x^{[n-1]}) + \sum_{j=2}^{n-1}\left(\frac{1}{n-1}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) + \frac{2}{n^2}\frac{\partial^2 m_{n-1}}{\partial x_2 \partial x_1}(x^{[n-1]})\right) +$$
$$\sum_{j=2}^{n-1}\left(\frac{n-3}{n^2}\frac{\partial^2 m_{n-1}}{\partial x_2 \partial x_1}(x^{[n-1]}) + \frac{1}{n^2}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\right) =$$
$$\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}),$$

which implies that

$$\frac{2(n-2)}{(n-1)^2}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) + \frac{4}{n^2}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) +$$
$$\frac{2(n-2)}{n^2}\frac{\partial^2 m_{n-1}}{\partial x_1 \partial x_2}(x^{[n-1]}) + (n-2)\frac{1}{n-1}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) +$$
$$(n-2)\left(\frac{n-1}{n^2}\frac{\partial^2 m_{n-1}}{\partial x_2 \partial x_1}(x^{[n-1]}) + \frac{1}{n^2}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})\right) =$$
$$\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}).$$

Hence

$$-\frac{-n^2+n+2}{(n-1)^2}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) + \frac{-n^2+n+2}{(n-1)^2}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) -$$
$$\frac{-n^2+n+2}{(n-1)^2}\frac{\partial^2 m_{n-1}}{\partial x_2 \partial x_1}(x^{[n-1]}) = 0,$$

which implies that $\frac{1}{(n-1)^2}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) - \frac{1}{n^2}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) + \frac{1}{n^2}\frac{\partial^2 m_{n-1}}{\partial x_2 \partial x_1}(x^{[n-1]}) = 0 \Rightarrow$
$\frac{1}{(n-1)^2}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) - \frac{n-1}{(n-2)n^2}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}) = 0 \Rightarrow$

$$\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]}) = \frac{(n-1)^3}{n^2(n-2)}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]}). \tag{3.33}$$

Letting $g = \frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}$ in (3.23) and replacing the index of summation, j, by l, we have

$$\frac{\partial}{\partial x_1}\left(\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}\right) = \sum_{l=1}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_j \partial x_i \partial x_l}(A)\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1}. \tag{3.34}$$

We now take $\frac{\partial}{\partial x_1}$ of both sides of (3.27) term by term. First,
$\frac{\partial}{\partial x_1}\left(\frac{\partial m_{n-1}}{\partial x_j}(A)\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2}\right) = \frac{\partial m_{n-1}}{\partial x_j}(A)\frac{\partial^3 m_n(\pi_{=j}\hat{x})}{\partial x_1^3} + \frac{\partial}{\partial x_1}\left(\frac{\partial m_{n-1}}{\partial x_j}(A)\right) \Rightarrow$ (by (3.26))

$$\frac{\partial}{\partial x_1}\left(\frac{\partial m_{n-1}}{\partial x_j}(A)\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2}\right) = \frac{\partial m_{n-1}}{\partial x_j}(A)\frac{\partial^3 m_n(\pi_{=j}\hat{x})}{\partial x_1^3} + \tag{3.35}$$
$$\sum_{i=1}^{n-1}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}.$$

Letting $g = \frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}$ in (3.23) and replacing the index of summation, j, by l, we have

$$\frac{\partial}{\partial x_1}\left(\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\right) = \sum_{l=1}^{n-1} \frac{\partial^3 m_{n-1}}{\partial x_j \partial x_i \partial x_l}(A)\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1}. \tag{3.36}$$

Second,

$$\frac{\partial}{\partial x_1}\left(\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\sum_{i=1}^{n-1}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}\right) =$$
$$\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\frac{\partial}{\partial x_1}\left(\sum_{i=1}^{n-1}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}\right) +$$
$$\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2}\sum_{i=1}^{n-1}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1},$$

which in turn equals

$$\sum_{i=1}^{n-1}(\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\left[\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial^2 m_n(\pi_{=i}\hat{x})}{\partial x_1^2} + \frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}\frac{\partial}{\partial x_1}(\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A))\right] +$$
$$\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}).$$

Hence, by (3.36), we have

$$\frac{\partial}{\partial x_1}\left(\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\sum_{i=1}^{n-1}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}\right) =$$
$$\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\sum_{i=1}^{n-1}(\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial^2 m_n(\pi_{=i}\hat{x})}{\partial x_1^2} + \frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}\sum_{l=1}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_j \partial x_i \partial x_l}(A)\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1} + \tag{3.37}$$
$$\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}).$$

Thus, adding (3.35) and (3.37) and taking $\frac{\partial}{\partial x_1}$ of both sides of (3.27), we have

$$\sum_{j=1}^{n-1}(\frac{\partial m_{n-1}}{\partial x_j}(A)\frac{\partial^3 m_n(\pi_{=j}\hat{x})}{\partial x_1^3} +$$
$$\sum_{i=1}^{n-1}(\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1} +$$
$$\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\left[\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial^2 m_n(\pi_{=i}\hat{x})}{\partial x_1^2} + \frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}\sum_{l=1}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_j \partial x_i \partial x_l}(A)\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1}\right] + \tag{3.38}$$
$$\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2}\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1})) = \frac{\partial^3 m_{n-1}}{\partial x_1^3}(\hat{x}).$$

Simplifying (3.38) a little bit yields

$$
\begin{gathered}
\sum_{j=1}^{n-1}\left(\frac{\partial m_{n-1}}{\partial x_j}(A)\frac{\partial^3 m_n(\pi_{=j}\hat{x})}{\partial x_1^3}+\right.\\
\sum_{i=1}^{n-1}\left(\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(A)\left(\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}+\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\frac{\partial^2 m_n(\pi_{=i}\hat{x})}{\partial x_1^2}+\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2}\right)\right.\\
\left.\left.+\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}\sum_{l=1}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_j \partial x_i \partial x_l}(A)\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1}\right)\right)=\frac{\partial^3 m_{n-1}}{\partial x_1^3}(\hat{x}).
\end{gathered}
\tag{3.39}
$$

Let $x_1 = \cdots = x_{n-1} = x$ in (3.39):

$$
\begin{gathered}
\sum_{j=1}^{n-1}\left(\frac{\partial m_{n-1}}{\partial x_j}(x^{[n-1]})\frac{\partial^3 m_n(\pi_{=j}\hat{x})}{\partial x_1^3}(x^{[n]})+\right.\\
\sum_{i=1}^{n-1}\left(\frac{\partial^2 m_{n-1}}{\partial x_j \partial x_i}(x^{[n]})\times\left(\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}(x^{[n]})+\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=i}\hat{x})}{\partial x_1^2}(x^{[n]})+\right.\right.\\
\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2}(x^{[n]})+\\
\left.\left.\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\left[\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}x^{[n]}\sum_{l=1}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_j \partial x_i \partial x_l}(x^{[n-1]})\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1}(x^{[n]})\right]\right)\right)\\
=\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]}).
\end{gathered}
\tag{3.40}
$$

First we need to break up the summation over j on the left hand side of (3.40):

$$
\begin{aligned}
&\sum_{j=2}^{n-1}\Big(\frac{\partial m_{n-1}}{\partial x_1}(x^{[n-1]})\frac{\partial^3 m_n(\pi_{=1}\hat{x})}{\partial x_1^3}(x^{[n]})+\\
&\frac{\partial m_{n-1}}{\partial x_j}(x^{[n-1]})\frac{\partial^3 m_n(\pi_{=j}\hat{x})}{\partial x_1^3}(x^{[n]})\Big)+\\
&\sum_{i=1}^{n-1}\Big(\frac{\partial^2 m_{n-1}}{\partial x_1\partial x_i}(x^{[n]})\times\Big(\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}(x^{[n]})+\\
&\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=i}\hat{x})}{\partial x_1^2}(x^{[n]})+\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=1}\hat{x})}{\partial x_1^2}(x^{[n]})\Big)\Big)+\\
&\sum_{j=2}^{n-1}\sum_{i=1}^{n-1}\Big(\frac{\partial^2 m_{n-1}}{\partial x_j\partial x_i}(x^{[n]})\times\Big(\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}(x^{[n]})+\\
&\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=i}\hat{x})}{\partial x_1^2}(x^{[n]})+\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2}(x^{[n]})\Big)+\\
&\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})\left[\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}x^{[n]}\sum_{l=1}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_1\partial x_i\partial x_l}(x^{[n-1]})\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1}(x^{[n]})\right]+\\
&\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\left[\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}x^{[n]}\sum_{l=1}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_j\partial x_i\partial x_l}(x^{[n-1]})\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1}(x^{[n]})\right]\Big)\Big).
\end{aligned}
\tag{3.41}
$$

Second, we need to break up the summation over i into $i=1, i=j$, and then $i\neq j$ in lines 3–8 of (3.41). We also need to break up the summation over l into $l=1, l\neq j, l\neq i$ in lines 4 and 5 of (3.41):

$$
\begin{aligned}
&\sum_{j=2}^{n-1}\frac{\partial m_{n-1}}{\partial x_1}(x^{[n-1]})\frac{\partial^3 m_n(\pi_{=1}\hat{x})}{\partial x_1^3}(x^{[n]})+\sum_{j=2}^{n-1}\frac{\partial m_{n-1}}{\partial x_j}(x^{[n-1]})\frac{\partial^3 m_n(\pi_{=j}\hat{x})}{\partial x_1^3}(x^{[n]})+\\
&\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n]})\left(\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})+2\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=1}\hat{x})}{\partial x_1^2}(x^{[n]})\right)+\\
&\sum_{i=2}^{n-1}\Big(\frac{\partial^2 m_{n-1}}{\partial x_1\partial x_i}(x^{[n]})\times\Big(\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}(x^{[n]})+\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=i}\hat{x})}{\partial x_1^2}(x^{[n]})+\\
&\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=1}\hat{x})}{\partial x_1^2}(x^{[n]})\Big)\Big)+\\
&\sum_{j=2}^{n-1}\Big(\frac{\partial^2 m_{n-1}}{\partial x_j\partial x_1}(x^{[n]})\times\Big(\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})+\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=1}\hat{x})}{\partial x_1^2}(x^{[n]})+\\
&\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2}(x^{[n]})\Big)\Big)+\\
&\frac{\partial^2 m_{n-1}}{\partial x_j^2}(x^{[n]})\left(\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})+2\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2}(x^{[n]})\right)+\\
&\sum_{\substack{i=2\\ i\neq j}}^{n-1}\Big(\frac{\partial^2 m_{n-1}}{\partial x_j\partial x_i}(x^{[n]})\times\Big(\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}(x^{[n]})+\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=i}\hat{x})}{\partial x_1^2}(x^{[n]})+\\
&\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial^2 m_n(\pi_{=j}\hat{x})}{\partial x_1^2}(x^{[n]})\Big)\Big)\Big)\Big)+\left(\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})\right)^3\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]})+\\
&\sum_{l=2}^{n-1}\Big(\frac{\partial^3 m_{n-1}}{\partial x_1^2\partial x_l}(x^{[n-1]})\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1}(x^{[n]})+
\end{aligned}
$$

$$\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}x^{[n]}\frac{\partial^3 m_{n-1}}{\partial x_1^2\partial x_i}(x^{[n-1]})\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})+$$
$$\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}x^{[n]}\frac{\partial^3 m_{n-1}}{\partial x_1\partial x_i^2}(x^{[n-1]})\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}(x^{[n]})+$$
$$\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}x^{[n]}(\sum_{\substack{l=2\\l\neq i}}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_1\partial x_i\partial x_l}(x^{[n-1]})\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1}(x^{[n]})))+$$
$$\sum_{j=2}^{n-1}(\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\sum_{i=1}^{n-1}(\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}x^{[n]}\sum_{l=1}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_j\partial x_i\partial x_l}(x^{[n-1]})\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1}(x^{[n]}))$$
$$+\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}x^{[n]}\frac{\partial^3 m_{n-1}}{\partial x_j\partial x_1^2}(x^{[n-1]})\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})+$$
$$\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}x^{[n]}\frac{\partial^3 m_{n-1}}{\partial x_j^2\partial x_1}(x^{[n-1]})\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})+$$
$$\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}x^{[n]}\left(\sum_{\substack{l=2\\l\neq j}}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_j\partial x_1\partial x_l}(x^{[n-1]})\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1}(x^{[n]})\right)+$$
$$\left(\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\right)^2\frac{\partial^3 m_{n-1}}{\partial x_j^2\partial x_1}(x^{[n-1]})\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})+$$
$$\left(\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\right)^3\frac{\partial^3 m_{n-1}}{\partial x_j^3}(x^{[n-1]})+$$
$$\left(\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\right)^2\sum_{\substack{l=2\\l\neq j}}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_j^2\partial x_l}(x^{[n-1]})\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1}(x^{[n]})+$$
$$\sum_{\substack{i=2\\i\neq j}}^{n-1}(\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}x^{[n]}\frac{\partial^3 m_{n-1}}{\partial x_j\partial x_i\partial x_1}(x^{[n-1]})\frac{\partial m_n(\pi_{=1}\hat{x})}{\partial x_1}(x^{[n]})+$$
$$\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\sum_{\substack{i=2\\i\neq j}}^{n-1}\left(\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}x^{[n]}\right)^2\frac{\partial^3 m_{n-1}}{\partial x_j\partial x_i^2}(x^{[n-1]})+$$
$$\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}x^{[n]}\frac{\partial^3 m_{n-1}}{\partial x_j^2\partial x_i}(x^{[n-1]})\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})+$$
$$\frac{\partial m_n(\pi_{=j}\hat{x})}{\partial x_1}(x^{[n]})\frac{\partial m_n(\pi_{=i}\hat{x})}{\partial x_1}x^{[n]}\sum_{\substack{l=2\\l\neq j,l\neq i}}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_j\partial x_i\partial x_l}(x^{[n-1]})\frac{\partial m_n(\pi_{=l}\hat{x})}{\partial x_1}(x^{[n]}))).$$

Now substitute and simplify using (3.28)–(3.30), Lemma 2, and Theorem 7:

$$\sum_{j=2}^{n-1}(\frac{1}{n-1}\left(2\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]})+6\frac{\partial^3 m_n}{\partial x_1^2\partial x_2}(x^{[n]})\right)+$$
$$\frac{1}{n-1}\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]})+\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n]})\left(\frac{2}{n}+\frac{4}{n}\left(\frac{2(n-2)}{n-1}\right)\right))+$$
$$\sum_{i=2}^{n-1}\left[\frac{\partial^2 m_{n-1}}{\partial x_1\partial x_2}(x^{[n]})\left(\frac{1}{n}+\frac{2}{n}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})+\frac{1}{n}\frac{2(n-2)}{n-1}\right)\right]+$$
$$\sum_{j=2}^{n-1}(\left[\frac{\partial^2 m_{n-1}}{\partial x_2\partial x_1}(x^{[n]})\left(\frac{2}{n}+\frac{1}{n}\frac{2(n-2)}{n-1}+\frac{2}{n}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\right)\right]+$$
$$\left[\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n]})\left(\frac{1}{n}+\frac{2}{n}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\right)\right]+$$
$$\sum_{\substack{i=2\\i\neq j}}^{n-1}\left[\frac{\partial^2 m_{n-1}}{\partial x_1\partial x_2}(x^{[n]})\left(\frac{1}{n}+\frac{1}{n}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})+\frac{1}{n}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\right)\right])+$$

$\frac{2}{n}\left[\frac{2}{n}\left(\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]})\frac{2}{n}+\sum_{l=2}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_1^2\partial x_2}(x^{[n-1]})\frac{1}{n}\right)\right]+$
$\frac{2}{n}\sum_{i=2}^{n-1}(\frac{\partial m_n}{\partial x_1}\left(x^{[n]}\right)\times(\frac{\partial^3 m_{n-1}}{\partial x_1^2\partial x_2}(x^{[n-1]})\frac{2}{n}+\frac{\partial^3 m_{n-1}}{\partial x_1\partial x_2^2}(x^{[n-1]})\frac{1}{n}+$
$\sum_{\substack{l=2\\l\neq i}}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_1\partial x_2\partial x_3}(x^{[n-1]})\frac{1}{n}))+$
$\sum_{j=2}^{n-1}(\frac{2}{n^2}\left(\frac{\partial^3 m_{n-1}}{\partial x_1^2\partial x_2}(x^{[n-1]})\frac{2}{n}+\frac{\partial^3 m_{n-1}}{\partial x_2^2\partial x_1}(x^{[n-1]})\frac{1}{n}+\sum_{\substack{l=2\\l\neq j}}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_3\partial x_1\partial x_2}(x^{[n-1]})\frac{1}{n}\right)+$
$\frac{1}{n^2}\left(\frac{\partial^3 m_{n-1}}{\partial x_2^2\partial x_1}(x^{[n-1]})\frac{2}{n}+\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]})\frac{1}{n}+\sum_{\substack{l=2\\l\neq j}}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_1^2\partial x_2}(x^{[n-1]})\frac{1}{n}\right)+$
$\frac{1}{n^2}\sum_{\substack{i=2\\i\neq j}}^{n-1}(\frac{\partial^3 m_{n-1}}{\partial x_2\partial x_3\partial x_1}(x^{[n-1]})\frac{2}{n}+\frac{\partial^3 m_{n-1}}{\partial x_1\partial x_2^2}(x^{[n-1]})\frac{1}{n}+$
$\frac{\partial^3 m_{n-1}}{\partial x_1^2\partial x_2}(x^{[n-1]})\frac{1}{n}+\sum_{\substack{l=2\\l\neq j,l\neq i}}^{n-1}\frac{\partial^3 m_{n-1}}{\partial x_1\partial x_2\partial x_3}(x^{[n-1]})\frac{1}{n})).$

A bit more simplification then yields

$$\begin{gathered}\frac{n}{n-1}\frac{\partial^3 m_n}{\partial x_1^3}(x^{[n]})+\frac{6}{n-1}\frac{\partial^3 m_n}{\partial x_1^2\partial x_2}(x^{[n]})+\frac{6+n}{n^3}\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]})+\\ \frac{3(n^2+n-6)}{n^3}\frac{\partial^3 m_{n-1}}{\partial x_1^2\partial x_2}(x^{[n-1]})+\frac{(n-3)(n-2)(n+2)}{n^3}\frac{\partial^3 m_{n-1}}{\partial x_1\partial x_2\partial x_3}(x^{[n-1]})+\\ \frac{2n^2+3n-15}{n(n-1)}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})+\\ \frac{2(n-2)(n^2-3)}{n(n-1)}\frac{\partial^2 m_n}{\partial x_1^2}(x^{[n]})\frac{\partial^2 m_{n-1}}{\partial x_1\partial x_2}(x^{[n-1]})=\frac{\partial^3 m_{n-1}}{\partial x_1^3}(x^{[n-1]}).\end{gathered}\tag{3.42}$$

Now substitute in (3.42) using $\frac{\partial^2 m_{n-1}}{\partial x_1\partial x_2}(x^{[n-1]})=-\frac{1}{n-2}\frac{\partial^2 m_{n-1}}{\partial x_1^2}(x^{[n-1]})$ from Theorem 7. That gives (2.8). ■

Proof of Proposition 1

Proof. Let $\hat{x}=(x_1,...,x_n)\,,W_n(\hat{x})=\sum_{k=1}^{n}w(x_k),v_k(\hat{x})=\frac{w(x_k)}{\sum_{k=1}^{n}w(x_k)}=\frac{w(x_k)}{W_n(\hat{x})}$, and $Q(\hat{x})=\sum_{k=1}^{n}v_k(\hat{x})h(x_k)=\frac{\sum_{k=1}^{n}h(x_k)w(x_k)}{\sum_{k=1}^{n}w(x_k)}$. We shall derive formulas for the partial derivatives of v_j and for Q, and finally for $m(\hat{x})=h^{-1}\left(Q(\hat{x})\right)$ itself. The proofs are a standard application of the product and quotient rules.

First Order

$\frac{\partial v_1(\hat{x})}{\partial x_1} = \frac{W_n(\hat{x})w'(x_1) - w(x_1)w'(x_1)}{(W_n(\hat{x}))^2} = w'(x_1)\frac{W_n(\hat{x}) - w(x_1)}{(W_n(\hat{x}))^2}$; Summarizing, we have

$$\begin{aligned}
\frac{\partial v_1(\hat{x})}{\partial x_1} &= w'(x_1)\frac{W_n(\hat{x}) - w(x_1)}{(W_n(\hat{x}))^2} \\
\frac{\partial v_1(\hat{x})}{\partial x_2} &= -\frac{w(x_1)w'(x_2)}{(W_n(\hat{x}))^2} \\
\frac{\partial v_k(\hat{x})}{\partial x_1} &= -\frac{w(x_k)w'(x_1)}{(W_n(\hat{x}))^2}, k \geq 2.
\end{aligned} \tag{3.43}$$

Second Order

$\frac{\partial^2 v_1(\hat{x})}{\partial x_1^2} = \left((W_n(\hat{x}))^2 \times (W_n(\hat{x})w''(x_1) + (w'(x_1))^2 - w(x_1)w''(x_1) - (w'(x_1))^2) - (W_n(\hat{x})w'(x_1) - w(x_1)w'(x_1))2W_n(\hat{x})w'(x_1)\right) / (W_n(\hat{x}))^4 \Rightarrow$

$$\begin{aligned}
\frac{\partial^2 v_1(\hat{x})}{\partial x_1^2} = &\left((W_n(\hat{x}))^2 w''(x_1) - W_n(\hat{x})(w(x_1)w''(x_1) + \right. \\
&\left. 2(w'(x_1))^2) + 2w(x_1)(w'(x_1))^2\right) / (W_n(\hat{x}))^3 .
\end{aligned} \tag{3.44}$$

$\frac{\partial^2 v_k(\hat{x})}{\partial x_1^2} = -w(x_k)\frac{(W_n(\hat{x}))^2 w''(x_1) - 2W_n(\hat{x})(w'(x_1))^2}{(W_n(\hat{x}))^4} = -w(x_k)\frac{W_n(\hat{x})w''(x_1) - 2(w'(x_1))^2}{(W_n(\hat{x}))^3}$

$\frac{\partial^2 v_k(\hat{x})}{\partial x_1^2} = -w(x_k) \times \left((W_n(\hat{x}))^2 w''(x_1) - 2W_n(\hat{x})(w'(x_1))^2\right) / (W_n(\hat{x}))^4 \Rightarrow$

$$\begin{aligned}
\frac{\partial^2 v_k(\hat{x})}{\partial x_1^2} = w(x_k) \times& \\
(2(w'(x_1))^2 - W_n(\hat{x})w''(x_1)) / (W_n(\hat{x}))^3 , k \geq 2.&
\end{aligned} \tag{3.45}$$

$\frac{\partial^2 v_1(\hat{x})}{\partial x_1 \partial x_2} = \left((W_n(\hat{x}))^2 w'(x_2)w'(x_1) - (W_n(\hat{x})w'(x_1) - w(x_1)w'(x_1))2W_n(\hat{x})w'(x_2)\right) / (W_n(\hat{x}))^4 \Rightarrow$

$$\frac{\partial^2 v_1(\hat{x})}{\partial x_1 \partial x_2} = w'(x_1)w'(x_2)\frac{-W_n(\hat{x}) + 2w(x_1)}{(W_n(\hat{x}))^3}. \tag{3.46}$$

$\frac{\partial^2 v_2(\hat{x})}{\partial x_1 \partial x_2} = (W_n(\hat{x})w'(x_1)w'(x_2) - 2(W_n(\hat{x})w'(x_2) - w(x_2)w'(x_2))w'(x_1)) / (W_n(\hat{x}))^3 \Rightarrow$

$$\frac{\partial^2 v_2(\hat{x})}{\partial x_1 \partial x_2} = w'(x_1)w'(x_2)\frac{-W_n(\hat{x}) + 2w(x_2)}{(W_n(\hat{x}))^3}. \tag{3.47}$$

$$\frac{\partial^2 v_k(\hat{x})}{\partial x_1 \partial x_2} = \frac{2w(x_k)w'(x_1)w'(x_2)}{(W_n(\hat{x}))^3}, k \geq 3. \tag{3.48}$$

Third Order

$\frac{\partial^3 v_1(\hat{x})}{\partial x_1^3} = (W_n(\hat{x}))^3 \times$
$\left((W_n(\hat{x}))^2 w'''(x_1) + 2W_n(\hat{x})w'(x_1)w''(x_1) - \right.$
$W_n(\hat{x})(w(x_1)w'''(x_1) + w'(x_1)w''(x_1) +$
$4w'(x_1)w''(x_1)) - w'(x_1)(w(x_1)w''(x_1) + 2(w'(x_1))^2) +$

$4w(x_1)w'(x_1)w''(x_1) + 2(w'(x_1))^3)/\,(W_n(\hat{x}))^6 -$
$(\,(W_n(\hat{x}))^2\, w''(x_1) - W_n(\hat{x})(w(x_1)w''(x_1)+$
$2(w'(x_1))^2) + 2w(x_1)(w'(x_1))^2) \times 3\,(W_n(\hat{x}))^2\, w'(x_1)/\,(W_n(\hat{x}))^6 =$
$W_n(\hat{x})\times$
$(\,(W_n(\hat{x}))^2\, w'''(x_1) + 2W_n(\hat{x})w'(x_1)w''(x_1) -$
$W_n(\hat{x})(w(x_1)w'''(x_1) + w'(x_1)w''(x_1)+$
$4w'(x_1)w''(x_1)) - w'(x_1)(w(x_1)w''(x_1) + 2(w'(x_1))^2)+$
$4w(x_1)w'(x_1)w''(x_1) + 2(w'(x_1))^3)/\,(W_n(\hat{x}))^4 -$
$(\,(W_n(\hat{x}))^2\, w''(x_1) - W_n(\hat{x})(w(x_1)w''(x_1)+$
$2(w'(x_1))^2) + 2w(x_1)(w'(x_1))^2) \times 3w'(x_1)/\,(W_n(\hat{x}))^4 \Rightarrow$

$$\begin{aligned}\frac{\partial^3 v_1(\hat{x})}{\partial x_1^3} &= (w'''(x_1)\,(W_n(\hat{x}))^3 - (6w'(x_1)w''(x_1) + w(x_1)w'''(x_1))\,(W_n(\hat{x}))^2 + \\ &\quad 6(w(x_1)w'(x_1)w''(x_1) + (w'(x_1))^3)W_n(\hat{x}) - 6w(x_1)(w'(x_1))^3)/ \\ &\quad (W_n(\hat{x}))^4\,. \end{aligned} \tag{3.49}$$

$\frac{\partial^3 v_k(\hat{x})}{\partial x_1^3} = w(x_k)\times(\,(W_n(\hat{x}))^3\times(4w'(x_1)w''(x_1) - W_n(\hat{x})w'''(x_1) - w'(x_1)w''(x_1)) -$
$(\,\left[2(w'(x_1))^2 - W_n(\hat{x})w''(x_1)\right] 3\,(W_n(\hat{x}))^2\, w'(x_1)))/\,(W_n(\hat{x}))^6 =$
$w(x_k) \times (W_n(\hat{x}) \times (4w'(x_1)w''(x_1) - W_n(\hat{x})w'''(x_1) - w'(x_1)w''(x_1)) -$
$3(\,\left[2(w'(x_1))^2 - W_n(\hat{x})w''(x_1)\right] w'(x_1)))/\,(W_n(\hat{x}))^4 \Rightarrow$

$$\begin{aligned}\frac{\partial^3 v_k(\hat{x})}{\partial x_1^3} = w(x_k) \times (\,-6(w'(x_1))^3 + 6W_n(\hat{x})w'(x_1)w''(x_1) \\ -\,(W_n(\hat{x}))^2\, w'''(x_1))/\,(W_n(\hat{x}))^4\,, k \geq 2.\end{aligned} \tag{3.50}$$

$\frac{\partial^3 v_1(\hat{x})}{\partial x_1^2 \partial x_2} = (W_n(\hat{x}))^3 \times$
$(2W_n(\hat{x})w'(x_2)w''(x_1) -$
$w'(x_2)(w(x_1)w''(x_1) + 2(w'(x_1))^2))/\,(W_n(\hat{x}))^6 -$
$3\,(W_n(\hat{x}))^2\, w'(x_2) \times (\,(W_n(\hat{x}))^2\, w''(x_1) -$
$W_n(\hat{x})(w(x_1)w''(x_1) + 2(w'(x_1))^2) + 2w(x_1)(w'(x_1))^2)/\,(W_n(\hat{x}))^6 =$
$w'(x_2)W_n(\hat{x})\times$
$(2W_n(\hat{x})w''(x_1) -$
$(w(x_1)w''(x_1) + 2(w'(x_1))^2))/\,(W_n(\hat{x}))^6 -$
$3(\,(W_n(\hat{x}))^2\, w''(x_1) -$
$W_n(\hat{x})(w(x_1)w''(x_1) + 2(w'(x_1))^2) + 2w(x_1)(w'(x_1))^2)/\,(W_n(\hat{x}))^4 \Rightarrow$

$$\begin{aligned}\frac{\partial^3 v_1(\hat{x})}{\partial x_1^2 \partial x_2} &= w'(x_2) \times (\,-(W_n(\hat{x}))^2\, w''(x_1) + 2W_n(\hat{x})(w(x_1)w''(x_1) + \\ &\quad 2(w'(x_1))^2) - 6w(x_1)(w'(x_1))^2)/\,(W_n(\hat{x}))^4\,.\end{aligned} \tag{3.51}$$

$\frac{\partial^3 v_2(\hat{x})}{\partial x_1^2 \partial x_2} = w(x_2) \times (\,(W_n(\hat{x}))^3\,[-w'(x_2)w''(x_1)] -$
$\left[2(w'(x_1))^2 - W_n(\hat{x})w''(x_1)\right] 3\,(W_n(\hat{x}))^2\, w'(x_2))/\,(W_n(\hat{x}))^6 +$
$w'(x_2) \times (2(w'(x_1))^2 - W_n(\hat{x})w''(x_1))/\,(W_n(\hat{x}))^3 =$

$w(x_2) \times ((W_n(\hat{x}))^3 [-w'(x_2)w''(x_1)] -$
$\left[2(w'(x_1))^2 - W_n(\hat{x})w''(x_1)\right] 3 (W_n(\hat{x}))^2 w'(x_2))/ (W_n(\hat{x}))^6 +$
$w'(x_2) \times (2(w'(x_1))^2 - W_n(\hat{x})w''(x_1))/ (W_n(\hat{x}))^3 =$
$-w(x_2) \times (W_n(\hat{x})w''(x_1)w(x_2)+$
$3 \left[2(w'(x_1))^2 - W_n(\hat{x})w''(x_1)\right] w(x_2))/ (W_n(\hat{x}))^4 +$
$w'(x_2) \times (2W_n(\hat{x})(w'(x_1))^2 - (W_n(\hat{x}))^2 w''(x_1))/ (W_n(\hat{x}))^4 \Rightarrow$

$$\begin{aligned} \frac{\partial^3 v_2(\hat{x})}{\partial x_1^2 \partial x_2} = w'(x_2) \times (- (W_n(\hat{x}))^2 w''(x_1)+ \\ 2W_n(\hat{x})(w''(x_1)w(x_2) + (w'(x_1))^2)- \\ 6(w'(x_1))^2 w(x_2))/ (W_n(\hat{x}))^4 . \end{aligned} \tag{3.52}$$

$\frac{\partial^3 v_k(\hat{x})}{\partial x_1^2 \partial x_2} = w(x_k) \times (- (W_n(\hat{x}))^3 w''(x_1)w'(x_2)-$
$\left[2(w'(x_1))^2 - W_n(\hat{x})w''(x_1)\right] 3 (W_n(\hat{x}))^2 w'(x_2))/ (W_n(\hat{x}))^6 =$
$-w(x_k) \times (W_n(\hat{x})w''(x_1)w'(x_2)+$
$6(w'(x_1))^2 w'(x_2) - 3W_n(\hat{x})w''(x_1)w'(x_2))/ (W_n(\hat{x}))^4 \Rightarrow$

$$\begin{aligned} \frac{\partial^3 v_k(\hat{x})}{\partial x_1^2 \partial x_2} = -2w(x_k)w'(x_2)\times \\ \frac{3(w'(x_1))^2 - W_n(\hat{x})w''(x_1)}{(W_n(\hat{x}))^4}, k \geq 3. \end{aligned} \tag{3.53}$$

Recall that for $x > 0$, we let $x^{[n]} = (x, ..., x) \in \Re^n_+$. Using $W_n(x^{[n]}) = nw(x)$ and substituting into (3.43)–(3.53), we have

First Order

$$\begin{aligned} \frac{\partial v_1}{\partial x_1}(x^{[n]}) &= \frac{(n-1)w'(x)}{n^2 w(x)}, \frac{\partial v_1}{\partial x_2}(x^{[n]}) = -\frac{w(x)w'(x)}{n^2 w^2(x)} = \\ -\frac{w'(x)}{n^2 w(x)}, \frac{\partial v_k}{\partial x_1}(x^{[n]}) &= -\frac{w'(x)}{n^2 w(x)}. \end{aligned} \tag{3.54}$$

Second Order

$\frac{\partial^2 v_1}{\partial x_1^2}(x^{[n]}) = (n^2 w^2(x)w''(x) - nw(x)(w(x)w''(x)+$
$2(w'(x))^2) + 2w(x)(w'(x))^2)/n^3 w^3(x) \Rightarrow$

$$\frac{\partial^2 v_1}{\partial x_1^2}(x^{[n]}) = \frac{(n-1)(nw(x)w''(x) - 2(w'(x))^2)}{n^3 w^2(x)}. \tag{3.55}$$

$\frac{\partial^2 v_k}{\partial x_1^2}(x^{[n]}) = w(x)\frac{2(w'(x))^2 - nw(x)w''(x)}{n^3 w^3(x)} \Rightarrow$

$$\frac{\partial^2 v_k}{\partial x_1^2}(x^{[n]}) = \frac{2(w'(x))^2 - nw(x)w''(x)}{n^3 w^2(x)}, k \geq 2. \tag{3.56}$$

$\frac{\partial^2 v_1}{\partial x_1 \partial x_2}(x^{[n]}) = (w'(x))^2 \frac{2w(x) - nw(x)}{n^3 w^3(x)} \Rightarrow$

$$\frac{\partial^2 v_1}{\partial x_1 \partial x_2}(x^{[n]}) = (w'(x))^2 \frac{2-n}{n^3 w^2(x)}. \tag{3.57}$$

$$\frac{\partial^2 v_2}{\partial x_1 \partial x_2}(x^{[n]}) = (w'(x))^2 \frac{2-n}{n^3 w^2(x)}, \frac{\partial^2 v_2}{\partial x_1 \partial x_2}(x^{[n]}) = \frac{2(w'(x))^2}{n^3 w^2(x)}, k \geq 3. \tag{3.58}$$

Third Order

$\frac{\partial^3 v_1}{\partial x_1^3}(x^{[n]}) = (w'''(x)n^3 w^3(x) - (6w'(x)w''(x) + w(x)w'''(x))n^2 w^2(x) + 6(w(x)w'(x)w''(x) + (w'(x))^3)nw(x) - 6w(x)(w'(x))^3)/n^4 w^3(x) \Rightarrow$

$$\begin{gathered}\frac{\partial^3 v_1}{\partial x_1^3}(x^{[n]}) = (n-1) \times (n^2 w^2(x) w'''(x) - \\ 6nw(x)w'(x)w''(x) + 6(w'(x))^3))/n^4 w^3(x).\end{gathered} \tag{3.59}$$

$\frac{\partial^3 v_1}{\partial x_1^2 \partial x_2}(x^{[n]}) = w'(x) \times (- n^2 w^2(x) w''(x) + 2nw(x)(w(x)w''(x) + 2(w'(x))^2) - 6w(x)(w'(x))^2)/n^4 w^4(x) \Rightarrow$

$$\begin{gathered}\frac{\partial^3 v_1}{\partial x_1^2 \partial x_2}(x^{[n]}) = w'(x) \times ((2n - n^2)w(x)w''(x) + \\ (4n-6)(w'(x))^2)/n^4 w^3(x).\end{gathered} \tag{3.60}$$

$\frac{\partial^3 v_2}{\partial x_1^2 \partial x_2}(x^{[n]}) = w'(x) \times (- n^2 w^2(x) w''(x) + 2nw(x)(w''(x)w(x) + (w'(x))^2) - 6(w'(x))^2 w(x))/n^4 w^4(x) \Rightarrow$

$$\begin{aligned}\frac{\partial^3 v_2}{\partial x_1^2 \partial x_2}(x^{[n]}) &= w'(x)\frac{(2n-n^2)w(x)w''(x) + (2n-6)(w'(x))^2}{n^4 w^3(x)} \\ \frac{\partial^3 v_k}{\partial x_1^2 \partial x_2}(x^{[n]}) &= 2w'(x)\frac{nw(x)w''(x) - 3(w'(x))^2}{n^4 w^3(x)}, k \geq 3.\end{aligned} \tag{3.61}$$

Write $Q(\hat{x}) = v_1(\hat{x})h(x_1) + \sum_{k=2}^{n} h(x_k)v_k(\hat{x}) \Rightarrow \frac{\partial Q}{\partial x_1}(\hat{x}) = v_1(\hat{x})h'(x_1) + \frac{\partial v_1(\hat{x})}{\partial x_1}h(x_1) + \sum_{k=2}^{n} h(x_k)\frac{\partial v_k(\hat{x})}{\partial x_1} \Rightarrow$

$$\frac{\partial Q}{\partial x_1}(x^{[n]}) = v_1(x^{[n]})h'(x) + h(x)\frac{\partial v_1}{\partial x_1}(x^{[n]}) + h(x)\sum_{k=2}^{n}\frac{\partial v_k}{\partial x_1}(x^{[n]}). \tag{3.62}$$

$$\frac{\partial^2 Q}{\partial x_1^2}(\hat{x}) = v_1(\hat{x})h''(x_1) + 2\frac{\partial v_1(\hat{x})}{\partial x_1}h'(x_1) + \frac{\partial^2 v_1(\hat{x})}{\partial x_1^2}h(x_1) + \sum_{k=2}^{n} h(x_k)\frac{\partial^2 v_k(\hat{x})}{\partial x_1^2} \Rightarrow$$

$$\frac{\partial^2 Q}{\partial x_1^2}(x^{[n]}) = h''(x)v_1(x^{[n]}) + 2h'(x)\frac{\partial v_1}{\partial x_1}(x^{[n]}) + h(x)\frac{\partial^2 v_1}{\partial x_1^2}(x^{[n]}) + h(x)\sum_{k=2}^{n}\frac{\partial^2 v_k}{\partial x_1^2}(x^{[n]}). \tag{3.63}$$

$\frac{\partial^2 Q}{\partial x_1 \partial x_2}(\hat{x}) = \frac{\partial}{\partial x_2}\left(v_1(\hat{x})h'(x_1) + \frac{\partial v_1(\hat{x})}{\partial x_1}h(x_1) + \sum_{k=2}^{n} h(x_k)\frac{\partial v_k(\hat{x})}{\partial x_1}\right) =$

$h'(x_1)\frac{\partial v_1(\hat{x})}{\partial x_2} + h(x_1)\frac{\partial^2 v_1(\hat{x})}{\partial x_1 \partial x_2} + h(x_2)\frac{\partial^2 v_2(\hat{x})}{\partial x_1 \partial x_2} + h'(x_2)\frac{\partial v_2(\hat{x})}{\partial x_1} + \sum_{k=3}^{n} h(x_k)\frac{\partial^2 v_k(\hat{x})}{\partial x_1 \partial x_2} \Rightarrow$

$$\begin{gathered}\frac{\partial^2 Q}{\partial x_1 \partial x_2}(x^{[n]}) = h'(x)\frac{\partial v_1}{\partial x_2}(x^{[n]}) + h(x)\frac{\partial^2 v_1}{\partial x_1 \partial x_2}(x^{[n]}) + h(x)\frac{\partial^2 v_2}{\partial x_1 \partial x_2}(x^{[n]}) + \\ h'(x)\frac{\partial v_2}{\partial x_1}(x^{[n]}) + h(x)\sum_{k=3}^{n}\frac{\partial^2 v_k}{\partial x_1 \partial x_2}(x^{[n]}).\end{gathered} \tag{3.64}$$

$\frac{\partial^3 Q}{\partial x_1^3}(\hat{x}) = v_1(\hat{x})h'''(x_1) + 3\frac{\partial v_1(\hat{x})}{\partial x_1}h''(x_1) + 3\frac{\partial^2 v_1(\hat{x})}{\partial x_1^2}h'(x_1)+$
$\frac{\partial^3 v_1(\hat{x})}{\partial x_1^3}h(x_1) + \sum_{k=2}^{n} h(x_k)\frac{\partial^3 v_k(\hat{x})}{\partial x_1^3} \Rightarrow$

$$\frac{\partial^3 Q}{\partial x_1^3}(x^{[n]}) = h'''(x)v_1(x^{[n]}) + 3h''(x)\frac{\partial v_1}{\partial x_1}(x^{[n]}) + 3h'(x)\frac{\partial^2 v_1}{\partial x_1^2}(x^{[n]})+ \quad (3.65)$$
$$h(x)\frac{\partial^3 v_1}{\partial x_1^3}(x^{[n]}) + h(x)\sum_{k=2}^{n}\frac{\partial^3 v_k}{\partial x_1^3}(x^{[n]}).$$

$\frac{\partial^3 Q}{\partial x_1^2 \partial x_2}(\hat{x}) = h''(x_1)\frac{\partial v_1(\hat{x})}{\partial x_2}+2h'(x_1)\frac{\partial^2 v_1(\hat{x})}{\partial x_1 \partial x_2} + h(x_1)\frac{\partial^3 v_1(\hat{x})}{\partial x_1^2 \partial x_2} + h(x_2)\frac{\partial^3 v_2(\hat{x})}{\partial x_1^2 \partial x_2} +$
$h'(x_2)\frac{\partial^2 v_2(\hat{x})}{\partial x_1^2} + \sum_{k=3}^{n} h(x_k)\frac{\partial^3 v_k(\hat{x})}{\partial x_1^2 \partial x_2} \Rightarrow$

$$\frac{\partial^3 Q}{\partial x_1^2 \partial x_2}(x^{[n]}) = h''(x)\frac{\partial v_1}{\partial x_2}(x^{[n]}) + 2h'(x)\frac{\partial^2 v_1}{\partial x_1 \partial x_2}(x^{[n]}) + h(x)\frac{\partial^3 v_1}{\partial x_1^2 \partial x_2}(x^{[n]})+ \quad (3.66)$$
$$h(x)\frac{\partial^3 v_2}{\partial x_1^2 \partial x_2}(x^{[n]}) + h'(x)\frac{\partial^2 v_2}{\partial x_1^2}(x^{[n]}) + h(x)\sum_{k=3}^{n}\frac{\partial^3 v_k}{\partial x_1^2 \partial x_2}(x^{[n]}).$$

Now substitute (3.54)–(3.61) into (3.62)–(3.66). That yields, after some simplification:

First Order: $\frac{\partial Q}{\partial x_1}(x^{[n]}) = \frac{1}{n}h'(x)$

Second Order: $\frac{\partial^2 Q}{\partial x_1^2}(x^{[n]}) = \frac{2(n-1)h'(x)w'(x)+nw(x)h''(x)}{n^2 w(x)}$,
$\frac{\partial^2 Q}{\partial x_1 \partial x_2}(x^{[n]}) = -2\frac{h'(x)(w(x))^2 w'(x)}{n^2(w(x))^3}$

Third order: $\frac{\partial^3 Q}{\partial x_1^3}(x^{[n]}) = (3n^2-3n)w(x)h'(x)w''(x)+$
$(3n^2-3n)w(x)h''(x)w'(x) + n^2\left(w(x)\right)^2 h'''(x)+$
$(6-6n)h'(x)(w'(x))^2)/n^3\left(w(x)\right)^2$
$\frac{\partial^3 Q}{\partial x_1^2 \partial x_2}(x^{[n]}) = \frac{(6-2n)h'(x)(w'(x))^2 - nw(x)h''(x)w'(x) - nw''(x)w(x)h'(x)}{n^3(w(x))^2}$

Finally, we obtain the partial derivatives of m using the formulas above for Q and the chain rule.

Second Order: $\frac{\partial^2 Q}{\partial x_1^2} = h'(m(\hat{x}))\frac{\partial^2 m}{\partial x_1^2} + h''(m(\hat{x}))\left(\frac{\partial m}{\partial x_1}\right)^2 \Rightarrow$
$\frac{\partial^2 Q}{\partial x_1^2}(x^{[n]}) = h'(x)\frac{\partial^2 m}{\partial x_1^2}(x^{[n]}) + \frac{1}{n^2}h''(x) \Rightarrow \frac{2(n-1)h'(x)w'(x)+nw(x)h''(x)}{n^2 w(x)} =$
$h'(x)\frac{\partial^2 m}{\partial x_1^2}(x^{[n]}) + \frac{1}{n^2}h''(x)$. Solving for $\frac{\partial^2 m}{\partial x_1^2}(x^{[n]})$ yields (2.1). One can derive (2.2) similarly or just use
(2.1) and Theorem 7(ii).

Third order: $\frac{\partial^2 Q}{\partial x_1^2}(\hat{x}) = h'(m(\hat{x}))\frac{\partial^2 m}{\partial x_1^2} + h''(m(\hat{x}))\left(\frac{\partial m}{\partial x_1}\right)^2 \Rightarrow$
$\frac{\partial^3 Q}{\partial x_1^3}(\hat{x}) = h'(m(\hat{x}))\frac{\partial^3 m}{\partial x_1^3}(\hat{x}) + h''(m(\hat{x}))\frac{\partial m}{\partial x_1}(\hat{x})\frac{\partial^2 m}{\partial x_1^2}(\hat{x})+$
$h''(m(\hat{x}))2\left(\frac{\partial m}{\partial x_1}(\hat{x})\right)\frac{\partial^2 m}{\partial x_1^2}(\hat{x}) + h'''(m(\hat{x}))\frac{\partial m}{\partial x_1}(\hat{x})\left(\frac{\partial m}{\partial x_1}(\hat{x})\right)^2 \Rightarrow$
$\frac{\partial^3 Q}{\partial x_1^3}(\hat{x}) = h'(m(\hat{x}))\frac{\partial^3 m}{\partial x_1^3}(\hat{x})+3h''(m(\hat{x}))\frac{\partial m}{\partial x_1}(\hat{x})\frac{\partial^2 m}{\partial x_1^2}(\hat{x})+h'''(m(\hat{x}))\left(\frac{\partial m}{\partial x_1}(\hat{x})\right)^3$

$\Rightarrow \frac{\partial^3 Q}{\partial x_1^3}(x^{[n]}) = h'(x)\frac{\partial^3 m}{\partial x_1^3}(x^{[n]}) + 3h''(x)\frac{1}{n}\frac{\partial^2 m}{\partial x_1^2}(x^{[n]}) + h'''(x)\frac{1}{n^3}$. Solving for $\frac{\partial^3 m}{\partial x_1^3}(x^{[n]})$ and using (2.1) yields (2.3).

$\frac{\partial^2 Q}{\partial x_1^2}(\hat{x}) = h'(m(\hat{x}))\frac{\partial^2 m}{\partial x_1^2}(\hat{x}) + h''(m(\hat{x}))\left(\frac{\partial m}{\partial x_1}(\hat{x})\right)^2 \Rightarrow$

$\frac{\partial^3 Q}{\partial x_1^2 \partial x_2}(\hat{x}) = h'(m(\hat{x}))\frac{\partial^3 m}{\partial x_1^2 \partial x_2}(\hat{x}) + h''(m(\hat{x}))\dfrac{\partial m}{\partial x_2}(\hat{x})\frac{\partial^2 m}{\partial x_1^2}(\hat{x}) +$

$h''(m(\hat{x}))2\frac{\partial m}{\partial x_1}(\hat{x})\frac{\partial^2 m}{\partial x_1 \partial x_2}(\hat{x}) + h'''(m(\hat{x}))\left(\frac{\partial m}{\partial x_1}(\hat{x})\right)^2 \dfrac{\partial m}{\partial x_2}(\hat{x}) \Rightarrow$

$\frac{\partial^3 Q}{\partial x_1^2 \partial x_2}(x^{[n]}) = h'(x)\frac{\partial^3 m}{\partial x_1^2 \partial x_2}(x^{[n]}) + \frac{1}{n}h''(x)\frac{\partial^2 m}{\partial x_1^2}(x^{[n]}) + \frac{2}{n}h''(x)\frac{\partial^2 m}{\partial x_1 \partial x_2}(x^{[n]}) + \frac{1}{n^3}h'''(x)$. Solving for $\frac{\partial^3 m}{\partial x_1^2 \partial x_2}(x^{[n]})$ and using (2.2) yields (2.4). ■

References

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